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Título de la tesis: Understanding the Benefits of Inflation-Linked Bonds: The Case of TIPS

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To my family

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"Happiness only real when shared." Christopher McCandless.

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Resumen

Los bonos indexados a la inflación son bonos en los que los pagos de intereses y el pago del principal se van ajustando por el incremento de los precios de bienes y servicios medidos por un índice de precios de referencia. En general se argumenta que este tipo de bonos genera beneficios para los distintos agentes de una economía. Desde el punto de vista de los inversores, los bonos indexados a la inflación permiten a los inversores tener un activo con el cual cubrirse ante el riesgo inflacionario. Desde el punto de vista de un gobierno, se argumenta que la emisión de este tipo de activo genera un ahorro en el coste de financiamiento ya que se ahorra el pago del premio por el riesgo inflacionario que está incluido en el rendimiento de los bonos nominales. Por último, los hacedores de políticas económicas pueden obtener información de mercado sobre las tasas de interés real a distintos plazos y sobre las expectativas de inflación. Este tipo de información es útil a la hora de la toma de decisiones.

Luego de muchos años de debate acerca de los beneficios de emitir deuda indexada a la inflación, el gobierno de EEUU comenzó a emitir deuda indexada a la inflación en 1997 bajo el nombre de "Treasury Inflation Protected Securities" o simplemente TIPS. Luego de 15 años de la primera emisión todavia no hay un acuerdo general sobre los beneficios empíricos del programa de emisión de los TIPS, ver Roush (2008), Dudley et al. (2009) y Fleckenstein et al. (2010). Esta tesis está dedicada a investigar acerca de los beneficios que generan los bonos indexados a la inflación en general y los beneficios generados por los TIPS en particular.

En el capítulo 2 se estudian los beneficios que los TIPS generan a los inversores. El aporte de este capítulo es medir empíricamente los beneficios de este tipo de bonos utilizando datos de mercado desde el año de su primer emisión en 1997 hasta el año 2010. Para comprender mejor y medir estos beneficios, el estudio se concentra en distintos atributos que pueden tener los inversores, como así también en las oportunidades de inversión que ellos pueden enfrentar. Se diferencian los tipos de inversores en función de su horizonte de inversión, de corto plazo y de largo plazo; y de acuerdo con su grado de aversión al riesgo. Luego, se mide el valor incremental que los TIPS proporcionan a los diferentes tipos de inversores en presencia de diversas combinaciones de clases de activos: renta variable, materias primas, y bienes raíces. Se demuestra que los inversores de largo plazo se ven beneficiados por los efectos de diversificación que ofrecen los TIPS, así como por la posibilidad de invertir en el activo libre de riesgo en términos reales. Mientras

tanto, los inversores de corto plazo pueden beneficiarse por la mejora en la frontera de oportunidades de inversión en términos reales.

En el capítulo 3 se desarrolla un modelo teórico para la estructura temporal de tasas de interés en el marco de la hipótesis de las preferencias definidas para estudiar las implicaciones sobre los beneficios de la emisión de bonos indexados a la inflación en lugar de lo bonos nominales convencionales. La hipótesis de las preferencias definidas, propuesto por Modigliani and Sutch (1966) y formalizado por Vayanos and Vila (2009), propone que los inversores tienen preferencias específicas sobre el vencimiento y tipo de bonos que desean. Sin embargo, existe un cierto grado de sustitución entre los distintos tipos bonos si la recompensa por cambiar de bono es lo suficientemente alta. La principal implicancia de esta hipótesis sobre la estructura temporal de tasas de interés es que los factores de demanda y oferta pueden afectar los rendimientos de los bonos, un aspecto que está ausente en los modelos de no arbitraje que están basados en el supuesto de un agente representativo.[1]

En este capítulo se extiende el trabajo realizado por Vayanos and Vila (2009) añadiendo inflación en el modelo y distinguiendo entre la estructura temporal de los tasas de interés reales y nominales. En cuanto a los beneficios de los bonos indexados: i) se encuentra que el premio por el riesgo inflacionario incluido en los rendimientos de los bonos nominales puede ser negativo, lo que implica que los bonos indexados a la inflación pueden generar una fuente de financiamiento más cara que los bonos nominales; ii) se plantean condiciones en las que la emisiones de bonos indexados a la inflación ayuda a un gobierno para conseguir diversificar la cartera de deuda; y iii) se obtiene que en períodos de dificultades financieras, la diferencia en los rendimientos nominales y reales no capturan adecuadamente las expectativas de inflación.

Una cuestión fundamental que se plantea en el estudio de los beneficios que genera la emisión de bonos indexados a la inflación es si el índice al que están indexados los bonos refleja con exactitud la exposición de los inversores a la inflación, la cual se define como "inflación de mercado". Cuando el grado de exposición de los inversores a los diferentes componentes de la inflación difiere de las ponderaciones utilizadas para calcular el índice que se utiliza para los bonos indexados, la inflación de mercado se diferenciará de la inflación del índice y el bono indexado ya no será el activo libre de riesgo en términos reales.

En el capítulo 4 se desarrolla y estima un modelo de no arbitraje de la estructura temporal de tasas de interés que ajusta los datos de bonos nominales emitidos por EEUU con el fin de obtener una medida de la inflación de mercado construida como la media ponderada de los principales componentes de la inflación medida por el índice de precios al consumidor (IPC). El objetivo es determinar si la inflación de mercado de EEUU difiere del índice utilizado en la indexación de los TIPS y determinar las implicacias sobre los beneficios de la emisión de bonos indexados a la inflación. La inflación de mercado

[1] Ver Cox et al. (1985), Duffie and Kan (1996), Duffee (2002) y Piazzesi (2005).

se estima como el promedio de la inflación subyacente, la inflación de alimentos y la inflación de energía, en donde los pesos difieren significativamente de los pesos fijados para la construcción del IPC, excepto el componente energético. Por lo tanto, se encuentra un riesgo de base para los inversores dado por la regla de indexación utilizada en los TIPS. También se encuentra que la diferencia en el premio por riesgo entre los bonos nominales y reales (indexados) con el mismo vencimiento no sólo es variable en el tiempo, sino también que su signo cambia. Esto implica que un gobierno no necesariamente va a ahorrar el premio por el riesgo de inflación mediante la emisión de bonos indexados en lugar de bonos nominales.

Chapter 1

Introduction

1.1 Introduction and Summary

Inflation-linked (IL) bonds are securities in which their cash flows are linked to any index that reflects fluctuations in prices of goods and services. They are mainly issued to protect investors' savings from inflation. IL bonds can be designed with a diverse cash flow and indexation structure, but generally, both interest and principal are linked to a price index so when prices go up interest payments and principal increase.[1] There are a variety of indexes to which IL bonds are linked such as consumer prices, retail prices, wholesale prices, commodities prices, and GDP deflator, among others. Although IL bonds can be issued by private entities, they are primarily issued by sovereign governments, such as U.S., U.K., Australia, Canada, among others.

The issuance of IL bonds, in general, is justified on welfare gains since it is argued that they provide benefits to the society.[2] From the investors' point of view, IL bonds can protect lenders against the erosion of their purchasing power and can serve as the real riskless asset for those investors that tend to hold the bonds until they mature. While a nominal bond with a certain term to mature offers a riskless nominal return to buy-and-hold investors, an IL bond with the same maturity provides a riskless real return to these buy-and-hold investors.[3]

From the Treasury's point of view, the main benefit of issuing this type of bond is that the government may reduce borrowing costs in two ways. First, since the real return of conventional nominal bonds is exposed to inflation risk, risk-averse investors will demand an inflation risk premium on nominal bonds' returns over IL bonds' returns. Thus, IL bonds will provide ex-ante cheaper funding than conventional nominal bond since the government will reduce debt-servicing by not having to pay the inflation risk premium included in nominal bonds' yields. Second, with IL bonds the government faces certainty about the real payments and uncertainty about the nominal payments it will be paying. Then, if realized inflation turns out to be lower than the market had expected at the time of issuance, IL bonds will provide ex-post cheaper funding than conventional nominal bond.

Moreover, from the policymakers' perspective the introduction of IL bonds can improve market information mechanisms and enhance the credibility of monetary policy. By having a well-developed market of nominal and IL bonds a monetary authority can extract instantaneous market information about real interest rate and inflation expectations over different horizons. This type of information is useful for policymakers since it helps them to interpret current conditions and to accurately forecast future economic activity. At the same time, the IL bonds issuance incentivizes the government to take an active

[1]See Chapter 2, Deacon et al. (2004), for further details about IL bonds' design.

[2]See for instance Shen (1995), Wrase (1997), Barr and Campbell (1997), Deacon et al. (2004), and Dudley et al. (2009).

[3]See Campbell and Viceira (2001), Brennan and Xia (2002), Campbell et al. (2003) and Wachter (2003).

role in controlling inflation since the higher realized inflation rate, the higher the funding costs for the Treasury.

After decades of debate the U.S. Treasury introduced IL bonds in the market in January 1997. They came in by the name of Treasury Inflation-Protected Securities (TIPS). Despite the fact that issuance of TIPS with different maturities has followed suit, there is no consensus among policymakers and academics on what particular benefits TIPS provide to the different stakeholders in the economy.[4] The presence of time-varying premiums (inflation and liquidity), and price pressures by institutional factors (demand and supply shocks) pose doubts regarding the theoretical benefits of IL bonds.[5] For instance, Fleckenstein et al. (2010) present the TIPS-Treasury bond puzzle by showing that TIPS are consistently undervalued relative to nominal U.S. Treasury bonds. In other words, they show that the U.S. Treasury has increased its financing cost by launching the TIPS program.[6]

In a conference held at the Federal Reserve Bank of New York (FRBNY) on February 10, 2009, leading academics, policymakers, and practitioners discussed about the cost and benefit of the TIPS program. Specifically, FRBNY President Dudley underlined the needs for an ex-ante analysis about potential benefits of issuing IL bonds apart from the difference in funding cost, such as diversification benefits of the Treasury's funding sources, access to a market-determined measure of inflation expectations, and the provision of a risk-free asset to long-term investors, for determining whether the decision to implement a TIPS program has been a good idea.[7] Along these lines, this thesis is devoted to studying the benefits of IL bonds, specifically to account for the benefits that TIPS have provided to the different stakeholders of the economy.

TIPS have been issued with maturities of 5, 10, 20, and 30 years, so they were primarily issued to provide a safe asset for investors with long investment horizon. The solid line in Figure 1.1 shows the evolution of TIPS' outstanding debt in billions (bn) of USD (indicated on left axis) while the dashed line (indicated on right axis) exhibits the evolution of TIPS' outstanding debt as a proportion of the total U.S. nominal outstanding debt issued with more than 2 years to mature (notes and bonds). TIPS' outstanding debt has increased at an average annual rate of 23% from \$33.0bn in 1997 to \$772.4bn in 2012 while its relative importance with respect to nominal debt has increased at an average annual rate of 26%

[4]See Roush (2008), Dudley et al. (2009) and Fleckenstein et al. (2010).

[5]Evidence of time-varying risk premia can be found in Sack and Elsasser (2004), Ang et al. (2008), D'Amico et al. (2010), Campbell et al. (2009) and Christensen and Gillan (2011). Greenwood and Vayanos (2010) provide evidence that the maturity structure of government debt affects bonds yields and excess returns.

[6]The common explanation for the TIPS-Treasury bond puzzle is because of a liquidity premium in TIPS, as explained in Sack and Elsasser (2004), Roush (2008), Dudley et al. (2009), D'Amico et al. (2010), Fleming and Krishnan (2009), Viceira and Pflueger (2011), among others. However, if IL bonds are bought by buy-and-hold investors with an investment horizon of the same length as the maturity of the bond, they should not be concerned about the poor liquidity in the secondary market of these instruments relative to nominal bonds, see Vayanos and Vila (1999).

[7]See http://www.newyorkfed.org/research/conference/2009/inflation/Summary_Conference_Findings.pdf.

from 1.23% in 1997 to 15.65% in 2008. After the 2008 financial crisis, the evolution of TIPS' debt as a proportion of the total U.S. nominal debt plunged and stabilized around 10%.

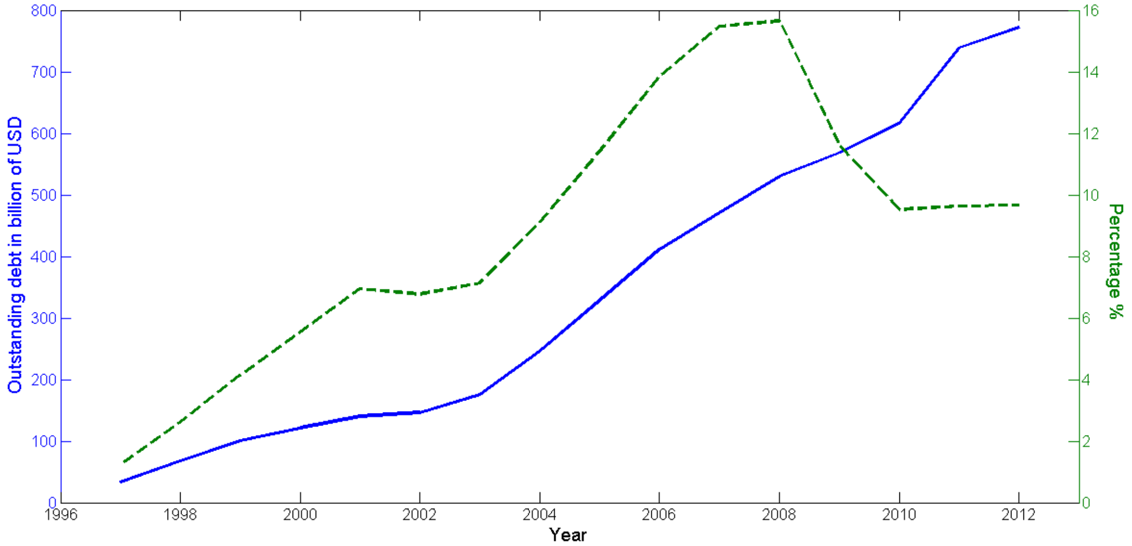


Figure 1.1: Evolution of TIPS growth

The main feature of TIPS is that their principal is linked to the U.S. non-seasonally adjusted consumer price index for all urban consumers (CPI-U). Thus, when CPI-U index goes up the principal of the TIPS is adjusted in order to capture the effect of inflation. TIPS pay interest twice a year at a fixed real rate applied to the adjusted principal and, at maturity, their final payment is given by the maximum value between the original principal and the adjusted principal. Table 1.1 exhibits an example of the TIPS' cash flow structure corresponding to the first TIPS issued on February 6, 1997 (coupon rate: 3.375%; maturity date: January 15, 2007). The second column of the Table exposes cash flow's dates while the third column shows the real cash flow amounts which are known in advance. The fourth column shows the index ratio at which the principal is adjusted due to inflation, and it is computed as the ratio between the reference CPI-U applicable to the settlement date divided by the reference CPI-U applicable to the original issue date of the bond. The fifth and sixth columns expose the inflation adjustment of each payoff and the final nominal cash flow amounts, respectively. The third and sixth columns show, in contrast to conventional nominal bonds, that TIPS have certain and constant payoffs in real terms while uncertain and variable payoffs in nominal terms.

In chapter 2, we address what benefits do TIPS provide to investors. We contribute to the literature by measuring the empirical benefits that TIPS provide to investors using market data that spans the period from the first issuance of TIPS in 1997 until 2010. To better understand and measure these benefits we focus on the different attributes of investors and on the investment opportunities they may have. We differentiate the types of investors according to their investment horizon, short-term and long-term, and

Table 1.1: TIPS' cash flow structure

Cash flow	Date	Real payment	Index ratio	Inflation adjustment	Nominal payment
Issue date	February 6, 1997	-$ 99.482	1.0000	$ 0.0000	-$ 99.482
First Interest Payment Date	July 15, 1997	$ 1.6875	1.0098	$ 0.0165	$ 1.7040
Interest Payment	January 15, 1998	$ 1.6875	1.0186	$ 0.0314	$ 1.7189
Interest Payment	July 15, 1998	$ 1.6875	1.0254	$ 0.0429	$ 1.7304
Interest Payment	January 15, 1999	$ 1.6875	1.0341	$ 0.0575	$ 1.7450
Interest Payment	July 15, 1999	$ 1.6875	1.0479	$ 0.0809	$ 1.7684
Interest Payment	January 15, 2000	$ 1.6875	1.0608	$ 0.1026	$ 1.7901
Interest Payment	July 15, 2000	$ 1.6875	1.0807	$ 0.1361	$ 1.8236
Interest Payment	January 15, 2001	$ 1.6875	1.0974	$ 0.1643	$ 1.8518
Interest Payment	July 15, 2001	$ 1.6875	1.1177	$ 0.1986	$ 1.8861
Interest Payment	January 15, 2002	$ 1.6875	1.1196	$ 0.2018	$ 1.8893
Interest Payment	July 15, 2002	$ 1.6875	1.1337	$ 0.2256	$ 1.9131
Interest Payment	January 15, 2003	$ 1.6875	1.1431	$ 0.2415	$ 1.9290
Interest Payment	July 15, 2003	$ 1.6875	1.1580	$ 0.2667	$ 1.9542
Interest Payment	January 15, 2004	$ 1.6875	1.1650	$ 0.2785	$ 1.9660
Interest Payment	July 15, 2004	$ 1.6875	1.1885	$ 0.3181	$ 2.0056
Interest Payment	January 15, 2005	$ 1.6875	1.2039	$ 0.3441	$ 2.0316
Interest Payment	July 15, 2005	$ 1.6875	1.2264	$ 0.3821	$ 2.0696
Interest Payment	January 15, 2006	$ 1.6875	1.2514	$ 0.4243	$ 2.1118
Interest Payment	July 15, 2006	$ 1.6875	1.2733	$ 0.4613	$ 2.1488
Principal plus Interest Payment	January 15, 2007	$ 101.688	1.2715	$ 27.6112	$ 129.299

according to their degree of risk aversion. Then, we measure the incremental value that TIPS provide to different types of investor in the presence of various combinations of asset classes: equity, commodities, and real estate. We show that while long-term investors can take advantage of the diversification effects that TIPS provide, as well as serving as the safe asset in their long-term investment problem, short-term investors may find them useful in improving the investment opportunity set in real terms. A shorter version of this chapter is forthcoming in the Journal of Empirical Finance and is coauthored with Ávaro Cartea and Juan Toro.

In Chapter 3, we develop a theoretical term structure model in a preferred-habitat framework to study the implications about the benefits of issuing IL bonds instead of conventional nominal ones. The preferred-habitat hypothesis, proposed by Modigliani and Sutch (1966) and formalized by Vayanos and Vila (2009), proposes that investors have specific maturity preferences but with some degree of substitution across types of bonds and maturity segments if the reward is high enough. The main implication of this hypothesis over the term structure of interest rates is that demand and supply factors affect bonds' yields which is an aspect that is absent in standard no-arbitrage term structure models that are based on the assumption of a representative agent model.[8]

We extend Vayanos and Vila (2009) by adding inflation into the model and by distinguishing between the real and the nominal term structure of interest rates. Regarding the benefits of IL bonds: i) we find that the inflation risk premium included in nominal bonds' yields might be negative implying that IL bonds can provide a more expensive

[8]See Cox et al. (1985), Duffie and Kan (1996), Duffee (2002) and Piazzesi (2005).

source of funding than conventional nominal bond; ii) we state some conditions under which the issuance of IL bonds will help the Treasury for diversification purposes; and, iii) we obtain that in periods of financial distress the difference in nominal and real yields may not adequately capture the compensation that buy-and-hold investors demand to cover expected inflation.

A fundamental question that arises when studying the benefits of issuing IL government bonds is whether the index to which IL bonds are linked accurately reflects inflation exposure of investors, which we define as the "market inflation". When the degree of investors' exposure to the different components of the CPI-U differs from the weights used to compute the true index, the market inflation will differ from CPI-U inflation and the IL bond will no longer be the real riskless asset.

In Chapter 4, we develop and estimate a no-arbitrage term structure model that fits U.S. nominal bonds data in order to obtain a measure of the U.S. expected market inflation as the weighted average of the main component of the CPI-U inflation. We address whether the U.S. expected market inflation differs from the inflation index used for TIPS and determine the main implications for the benefits of issuing IL bonds. We estimate the market inflation as the weighted average of the expected core inflation, food inflation, and energy inflation in which all weights significantly differ from the weights of the CPI-U inflation except the energy component. We find an inflation basis risk included in TIPS indexation rule due to the fact that market inflation differs from the inflation index to which TIPS are linked. We also find that the difference in premium between nominal and real bonds with the same maturity is not only time-varying but also the sign changes through time which implies that a government will not necessarily save the inflation risk premium by issuing IL bonds instead of nominal ones.

In Chapter 5, we summarize and expose the main findings of this thesis and discuss future lines of research.

Chapter 2

Optimal Portfolio Choice in Real Terms: Measuring the Benefits of TIPS

2.1 Introduction

Over the few last decades, a large number of articles by academics and practitioners have examined the arguments for and against issuing inflation indexed securities. Treasury Inflation-Protected Securities (TIPS) are a particular type of indexed bonds which are issued by the U.S. Treasury who introduced them to the market in January 1997. Despite the fact that issuance of TIPS with different maturities has followed suit, there is no consensus among policymakers and academics on what particular benefits TIPS provide to the different stakeholders in the economy, see Dudley et al. (2009) and Fleckenstein et al. (2010).

In general, it is argued that inflation protected government bonds provide benefits to the Treasury, policymakers, and investors, see for example Shen (1995), Barr and Campbell (1997), and Deacon et al. (2004). From the Treasury's point of view, the main benefit of issuing this type of bond is that they may reduce borrowing costs by not having to pay the inflation risk premium. From the policymakers perspective, it is argued that by introducing inflation linked bonds, they can improve market information mechanisms and enhance the credibility of the monetary policy because their issuance incentivizes the government to take an active role in controlling inflation. Finally, from the investors' point of view, inflation-indexed bonds can protect lenders against the erosion of their purchasing power.

The main question we address in this chapter is what benefits do TIPS provide to investors. To better understand and measure these benefits we focus on the different attributes of investors and on the investment opportunities they may have. We differentiate the types of investors according to their investment horizon, short-term and long-term, and according to their degree of risk aversion. Then, we measure the incremental value that TIPS provide to different types of investor in the presence of various combinations of asset classes: equity, commodities, and real estate.

We consider both long and short-term investors since the time horizon over which the investor plans to hold TIPS is relevant, due to the fact that TIPS only offer full protection against inflation if held until maturity.[1] TIPS have been issued with maturities of 5, 10, 20, and 30 years which makes them the riskless asset in real terms (a perfect hedge against inflation) for buy-and-hold long-term investors whose investment horizon perfectly matches the maturity of a TIPS. On the other hand, short-term investors see TIPS as "risky" assets, both in nominal and real terms because changes in expected real rates affect TIPS' returns. Moreover, when real interest rates rise, investors who purchased TIPS will suffer a capital loss in greater proportion than those who purchased conventional bonds

[1]There is some minimal inflation basis risk included in TIPS due to the fact that: a) the investor's basket might differ from the basket used to calculate the CPI-U to which the TIPS is indexed; b) there is a three month lag in the indexation rule; c) there are tax considerations; and, d) there is reinvestment risk arising from the coupon flows received before maturity.

with the same maturity.[2] At the same time however, short-term investors may benefit from the introduction of TIPS if they can improve the investment efficient frontier to increase the returns per unit of risk.

This chapter solves an optimal portfolio choice problem in real terms in order to measure the benefits of TIPS from the investor's point of view. We assume that the investor's strategy consists of finding the optimal allocation over a fixed horizon without rebalancing at intermediate points in time.[3] One of the empirical issues that the model handles is the possible mismatch between the investor's horizon and the maturity of TIPS. Since the first issuance in 1997 until now, the maturity of most outstanding TIPS at any point in time has been larger than one year. Moreover, off-the-run TIPS with less than one year to maturity are not easy to find in the secondary market which results in extremely high transaction costs. Thus, only buy-and-hold long-term investors have been able to lend at the risk-free rate in real terms. The model shows that short-term investors deal with uncertainty about inflation through the covariances between the returns of risky assets (one of which is TIPS) and inflation. In general, it is useful to distinguish the nominal from the real optimal portfolio choice problem if there is uncertainty about future inflation rates and there is a riskless asset in real terms, or when there is uncertainty about future inflation rates and assets in the investment opportunity set covary with inflation.

This chapter contributes to the literature by measuring the empirical benefits that TIPS provide to investors using market data that spans the period from the first issuance of TIPS in 1997 until 2010. We show that while long-term investors can take advantage of the diversification effects that TIPS provide, as well as serving as the safe asset in their long-term investment problem, short-term investors may find them useful in improving the investment opportunity set in real terms. We summarize some of our findings according to the investment horizon of the investor: short-term and long-term.

For short-term investors we highlight four empirical findings. First, we find that risk averse short-term investors who are not affected by money illusion find it optimal to replace part of their investment in long-term nominal bonds with TIPS for two reasons. One, TIPS yield a slightly higher average return than nominal bonds, and two, the covariance of TIPS' returns with inflation is higher than the covariance of the returns of nominal bonds with inflation. Second, the positive correlation of TIPS' nominal returns with inflation makes TIPS desirable for highly risk averse investors since they can be used to reduce the portfolio variance in real terms. Third, although the relative benefits from the introduction of TIPS diminish when the short-term investor has a wider investment opportunity set which might include gold, commodities, or real estate, highly risk averse investors still devote a fraction of their wealth to TIPS. Interestingly, when commodities

[2]TIPS have longer duration than nominal T-bonds with the same maturity. Broadly speaking, the duration of a bond is the length of time before the bond is due to be repaid. Thus, it is a measure of the bond's price sensitivity to interest rate movements.

[3]Brennan and Xia (2002) report that when the investor faces short-selling constraints the optimal myopic strategy is close to the dynamic strategy which includes a hedging demand term.

are available, the improvement to highly risk averse investors decreases because commodities are a better hedge against inflation than TIPS.[4] Finally, investors characterized by low levels of risk aversion do not obtain any benefit from the introduction of TIPS when there is a wider investment opportunity set that includes: stocks, nominal bonds, commodities, real estate, and the short-term nominal riskless asset (T-bill).

For buy-and-hold long-term investors we highlight four empirical findings. First, infinitely risk averse investors who are not affected by money illusion allocate all their wealth to the risk-free asset in real terms, as predicted by the theoretical model of Wachter (2003). Second, for all levels of relative risk aversion, nominal bonds are crowded out by TIPS. Third, when real estate is part of the investment opportunity set, the relative benefits from TIPS diminish because real estate's expected real return, corrected by risk, is high enough to outperform the real yield of TIPS. Finally, investors characterized by a log utility function do not obtain any benefits from the introduction of TIPS.

The remainder of this chapter is structured as follows. Section 2.2 looks at the existing literature on TIPS and summarizes previous findings on the potential benefits of TIPS for different types of investors. Section 2.3 derives the model that we employ to analyze the benefits of TIPS. Section 2.4 presents TIPS data and discusses its statistical properties since their first issuance in 1997 until 2010. Section 2.5 presents the empirical results and Section 2.6 concludes.

2.2 Literature Review

The literature on investment allocation and indexed bonds is too large to be covered here, so we focus on articles that are directly relevant to the subject matter of our study. We summarize the main findings of papers that have: discussed inflation as a variable which affects asset allocation; and studied the empirical benefits that TIPS provide to short-term investors.

The seminal work of Markowitz (1952) provides a mean-variance framework for asset allocation. This analysis has been followed by a large number of studies that have stressed different aspects of the portfolio allocation problem. Some of the extensions that appeared in the 1970s and 1980s introduce inflation as a relevant variable (see for example Sarnat (1973), Biger (1975), Lintner (1975), Friend et al. (1976), Solnik (1978), and Levy and Levy (1987)). The most significant result of these studies is that when there is uncertainty about future inflation the riskless asset should be a one-period inflation-linked bond.

More recent articles employ the mean-variance analysis to measure the empirical ben-

[4]Notwithstanding, the fraction of wealth invested in commodities is low for all levels of risk aversion underlying the fact that although commodities benefit from unexpected spikes of inflation they do not provide a reliable hedge in real terms due to the high volatility of their returns. We are grateful to an anonymous referee for pointing this out.

efits of TIPS for short-term investors, e.g. Lucas and Quek (1998), Kopcke and Kimball (1999), Roll (2004), Kothari and Shanken (2004), Hunter and Simon (2005), Brière and Signori (2009), among others. The conclusions of these studies vary depending on: i) the number of assets considered in the investment opportunity set; ii) the investment horizon employed in the calculations; iii) amount of data employed; or iv) assumptions made to compute returns. Moreover, apart from Kothari and Shanken (2004), they all assume that investors make allocation decisions in nominal terms. That is, investors are not worried about the purchasing power of their terminal wealth.

The early study of Kopcke and Kimball (1999), which uses only two years of TIPS data, finds that in periods of low and falling rates of inflation, short-term investors with any level of risk aversion decide not to invest in TIPS; and the optimal allocation consists of a mix of T-bills, conventional nominal bonds, and stocks. Furthermore, they find that the only scenario under which TIPS are included in the optimal portfolio is when investors are highly risk averse and they cannot invest in T-bills. Our study, however, finds that investors with any degree of risk aversion (apart from log utility) will include TIPS in the optimal portfolio which also contains T-bills, conventional nominal bonds, and stocks.

Hunter and Simon (2005) use conditional mean-variance spanning tests to provide evidence that TIPS do not provide statistically significant diversification benefits to short-term investors that hold cash, nominal bonds, and equities. In the same vein, Brière and Signori (2009) show that the combined effect of stable expectations of inflation rates and the increase in the liquidity of TIPS results in a decreasing ability of TIPS to provide diversification effects in the portfolio due to their high correlation with nominal bonds. Our findings contradict these results and this is due to the diversification role that TIPS play in an investor's allocation problem. And more generally, in light of the results of this chapter, the conclusions of these two previous studies could result from assuming that investors might suffer from money illusion.

Among the studies which find that TIPS provide benefits to investors is that of Roll (2004) who looks at the correlations of TIPS' returns with the returns of nominal bonds and equity between January 1997 and September 2003. Roll (2004) finds that TIPS improve the investment efficient frontier for short-term investors which is consistent with our empirical results. Kothari and Shanken (2004) study the optimal portfolio implications when both real and nominal returns are considered. They conclude that in an efficient portfolio with a one-year investment horizon and with assets restricted to stocks and bonds, substantial weight should be given to indexed bonds. These findings are confirmed in this chapter where we obtain that risk-averse short-term investors find it optimal to replace part of their investment in long-term nominal bonds with TIPS. This could be explained by two complementary arguments. First, TIPS yield a slightly higher average return than nominal bonds, and second, the positive correlation of TIPS' nominal returns with the rate of inflation reduces the variance of the portfolio real returns.

Kothari and Shanken (2004) also speculate that: "It will be interesting to see whether this conclusion persists when allocations over longer horizons and across a broader range of assets, including global equities and bonds, are examined in future research." Our study fills this gap and finds that when the short-term investor has a wider investment opportunity set which might include gold, commodities, or real estate, highly risk averse investors still devote a fraction of their wealth to TIPS even though the incremental benefits from their introduction decrease.

The academic literature recognizes that TIPS are the safest asset for buy-and-hold long-term investors, but to the best of our knowledge this is the first study that investigates the empirical benefits of their introduction. We establish empirically that for buy-and-hold long-term investors, nominal bonds are dominated by TIPS for all levels of risk aversion. Previous work has looked at the benefits from introducing in the investment opportunity set assets that yield a real return. For example, Campbell et al. (2003) show that an infinitely-lived investor with Epstein-Zin preferences greatly benefits from the addition of a (hypothetical) real perpetuity or consol to his investment opportunity set that includes nominal bonds and stocks.

Among the theoretical papers that study the role of bonds that protect the bearer against inflation are Campbell and Viceira (2001), Campbell et al. (2003), Brennan and Xia (2002), Illeditsch (2009) and Wachter (2003). Campbell and Viceira (2001) and Campbell et al. (2003) show that in a world where investment opportunities are time-varying, an inflation-indexed perpetuity bond is the riskless asset for infinite-lived investors who care about the stream of consumption in every period. Similarly, Brennan and Xia (2002) develop an optimal dynamic portfolio problem for a finite-lived investor who is able to invest in stocks or nominal bonds. They show that an infinitely risk averse investor, who is unconstrained to take short positions, invests in a mix of nominal bonds to replicate the return of an inflation indexed bond with maturity equal to the remaining investment horizon. Illeditsch (2009) extends previous work and includes inflation protected bonds in the analysis. He finds that the real instantaneously risk-free asset can be obtained with a long position in inflation-protected bonds, and a zero-investment portfolio of nominal bonds together with the nominal money market account. Finally, Wachter (2003) formalizes the "preferred habitat" hypothesis of Modigliani and Sutch (1966) and shows that investors who keep their investment profile fixed for a known length of time, will consider a bond with maturity equal to their investment period as the riskless asset. In agreement with Wachter (2003) we find that buy-and-hold long-term investors who are infinitely risk averse and are not affected by money illusion allocate all their wealth to the risk-free asset.

2.3 The Model: One-Period Portfolio Choice with Inflation

The classical mean-variance analysis for portfolio selection is usually posed in nominal terms, and if inflation is considered, the general approach implicitly assumes one of the following: i) investors suffer from money illusion; ii) the conditional variance of the inflation rate is zero; iii) there is no riskless real asset and assets' nominal returns are uncorrelated with inflation.

These three assumptions may not hold in practice and will adversely affect the portfolio allocation of investors that wish to protect their wealth from inflation. First, although investors may suffer from money illusion,[5] the potential benefits stemming from the introduction of a new asset which may be correlated with inflation should be measured in a framework stated in real terms. Otherwise the benefits may be under or overestimated. Second, certainty about future inflation rates may hold when the investment horizon is very short, but it is untenable for a long investment horizon. Finally, the introduction of TIPS allows buy-and-hold long-term investors to lend at a real riskless rate and empirical evidence rejects the assumption of independence between inflation and assets' nominal returns.[6]

2.3.1 The model

In this section, we expose an optimal portfolio choice problem for investors who consider the effects of inflation in their investment holdings. Intuitively, investors can avoid exposure to inflation risk by investing in a riskless asset in real terms and/or by investing in assets that covary with inflation. Therefore, we solve the investment allocation problem for investors both with and without a risk-free asset in real terms. We consider the optimal investment allocation of investors who are not worried about what may happen beyond the immediate next period and care about the purchasing power of their wealth.[7]

We assume that investors have a power utility function, defined over terminal real wealth and characterized by the Arrow-Pratt relative risk aversion coefficient γ. The investor is concerned with maximizing terminal wealth by choosing the optimal investment portfolio:

$$\max \quad \mathrm{E}_t\left[\frac{W_{t+1}^{1-\gamma}}{(1-\gamma)}\right] \tag{2.1}$$

[5]Cohen et al. (2005) provide empirical evidence to support that the stock market suffers from money illusion.

[6]Several articles in the 70s report evidence of negative correlation between nominal stock returns and inflation for short-term horizons; Bodie (1976), Nelson (1976), Fama and Schwert (1977), Jaffe and Mandelker (1976), among others. At long-horizons, Boudoukh and Richardson (1993) find that nominal stock returns are positively related (ex-ante and ex-post) with inflation. Schotman and Schweitzer (2000) show that stocks can be a hedge against inflation for different investment horizons where the crucial parameter for the results is the persistence of inflation. Roache and Attie (2009) provide a detailed literature review about the properties of different assets to hedge inflation.

[7]We assume that all investors are price takers, and that there are no taxes or transaction costs.

subject to the budget constraint

$$W_{t+1} = \big(1 + R_{p,t+1}\big)W_t\,. \tag{2.2}$$

The terminal real wealth W_{t+1} is equal to the initial real wealth W_t invested in the portfolio plus the portfolio's real return $R_{p,t+1}$. In order to simplify notation, we omit the time subindex from this point onwards.

Portfolio choice with a riskless real asset. Under the assumption that the portfolio's real return is lognormally distributed, it can be shown that the one-period optimization problem for an investor in real terms when there is a riskless real asset is[8]

$$\max_{\boldsymbol{\alpha}} \quad \boldsymbol{\alpha}'\mathrm{E}\big[\mathbf{r} - r_f\mathbf{1}\big] + \frac{1}{2}\boldsymbol{\alpha}'\boldsymbol{\sigma}^2 - \frac{\gamma}{2}\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha} \tag{2.3}$$

where $\mathrm{E}\big[\mathbf{r} - r_f\mathbf{1}\big]$ denotes an $n \times 1$ vector of risky assets' expected excess log real return over the log real riskless rate;[9] $\boldsymbol{\alpha}$ is a column vector with the allocation of the n risky assets; $\mathbf{1}$ is an $n \times 1$ column vector of ones; $\boldsymbol{\Sigma}$ represents the variance-covariance matrix of log real returns in which $\boldsymbol{\sigma}^2$ is $n \times 1$ vector of variances of real log asset returns.

The solution of the maximization problem in (2.3) is

$$\boldsymbol{\alpha}^* = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\mathrm{E}\big[\mathbf{r} - r_f\mathbf{1}\big] + \frac{\boldsymbol{\sigma}^2}{2}\right), \tag{2.4}$$

where the optimal allocation in the real risk-free assets is $1 - \mathbf{1}'\boldsymbol{\alpha}^*$. Imposing short-selling constraints in problem (2.3):

$$\alpha_i^* = 0 \qquad \Rightarrow \qquad \mathrm{E}\big[r_i\big] + \frac{1}{2}\sigma_{r_i}^2 - \gamma\sum_{\substack{j=1\\ j\neq i}}^{n-1}\alpha_j^*\sigma_{(i,j)} < r_f \tag{2.5}$$

$$\mathbf{1}'\boldsymbol{\alpha}^* = 1 \quad \Rightarrow \qquad \mathrm{E}\big[r_i\big] + \frac{1}{2}\sigma_{r_i}^2 - \gamma\sum_{j=1}^{n}\alpha_j^*\sigma_{(i,j)} > r_f \quad \text{for all } i : \alpha_i^* \geq 0. \tag{2.6}$$

Equation (2.5) shows the case when the short-selling restriction for a risky asset is binding. If the expected real return of asset i, corrected by risk, is lower than the real risk-free rate, the investor would like to short-sell it. However, since short-selling is not allowed, the investor sets his holdings in that particular asset to zero. Note that the second term in the left-hand side (LHS) corresponds to Jensen's correction of the log function, while

[8]See Campbell and Viceira (2002) for details. The problem is a discrete approximation to the continuous-time case. Campbell and Viceira (2002) remark that short horizon effects appear when the approximation is applied over long holding periods. Gil-Bazo (2006) shows that the approximate solution performs remarkably well under the stationary assumption, even for a long investment horizon. The one-period optimization problem can easily be generalized for the case of buy-and-hold investors with longer investment horizons, see Campbell and Viceira (2004).

[9]For simplicity, we label it in this way even when it is the log of one plus the portfolio's return.

the third one is the risk aversion correction.

Equation (2.6) restricts the investor from borrowing money at the real risk-free rate; investors are not able to issue inflation-linked bonds. The investor would like to borrow money at the real risk-free rate when the expected real return of assets, corrected by risk, is higher than the real risk-free rate. Investors will benefit from the introduction of the real riskless asset if they are able to borrow at that rate or if the real interest rate is high enough.

Portfolio choice without a riskless real asset. If investors are not able to find a riskless real asset the one-period optimization problem in real terms becomes

$$\max_{\boldsymbol{\alpha}} \quad \boldsymbol{\alpha}'\mathrm{E}\big[\mathbf{r} - r_0\mathbf{1}\big] + \frac{1}{2}\boldsymbol{\alpha}'\boldsymbol{\sigma}^2 - \frac{\gamma}{2}\boldsymbol{\alpha}'\boldsymbol{\Sigma}\boldsymbol{\alpha} + (1-\gamma)\boldsymbol{\alpha}'\boldsymbol{\sigma}_0 \tag{2.7}$$

where r_0 denotes the real return of a risky benchmark asset and $\boldsymbol{\sigma}_0$ is an $n \times 1$ vector of covariances of excess log real returns with the benchmark log return. In this case the vector $\boldsymbol{\sigma}^2$ and the matrix $\boldsymbol{\Sigma}$ contain variances and covariances of excess log real returns on the risky assets over the benchmark asset. Since our benchmark asset is the one-period nominal riskless rate (a risky asset in real terms) the solution to (2.7) is

$$\boldsymbol{\alpha}^{**} = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\left(\mathrm{E}\big[\mathbf{r} - r_0\mathbf{1}\big] + \frac{\boldsymbol{\sigma}^2}{2}\right) + \left(1 - \frac{1}{\gamma}\right)\boldsymbol{\Sigma}^{-1}\boldsymbol{\sigma}_{r\pi}\,, \tag{2.8}$$

where $\boldsymbol{\sigma}_{r\pi}$ denotes a column vector with covariances between excess log real returns and inflation.[10] In the absence of a real risk-free asset, investors deal with uncertainty about inflation through the covariances between the returns of risky assets and inflation. Securities which are correlated with inflation help to hedge against inflation risk, reducing the portfolio variance in real terms. Imposing short-selling constraints:

$$\alpha_i^{**} = 0 \qquad \Rightarrow \quad \mathrm{E}\big[r_i\big] + \frac{1}{2}\sigma_{r_i}^2 + (\gamma-1)\sigma_{r_i\pi} - \gamma\sum_{\substack{j=1\\ j\neq i}}^{n-1}\alpha_j^{**}\sigma_{(i,j)} < r_0 \tag{2.9}$$

$$\mathbf{1}'\alpha^{**} = 1 \quad \Rightarrow \quad \mathrm{E}\big[r_i\big] + \frac{1}{2}\sigma_{r_i}^2 + (\gamma-1)\sigma_{r_i\pi} - \gamma\sum_{j=1}^{n}\alpha_j^{**}\sigma_{(i,j)} > r_0, \quad \text{for all } i : \alpha_i^{**} \geq 0. \tag{2.10}$$

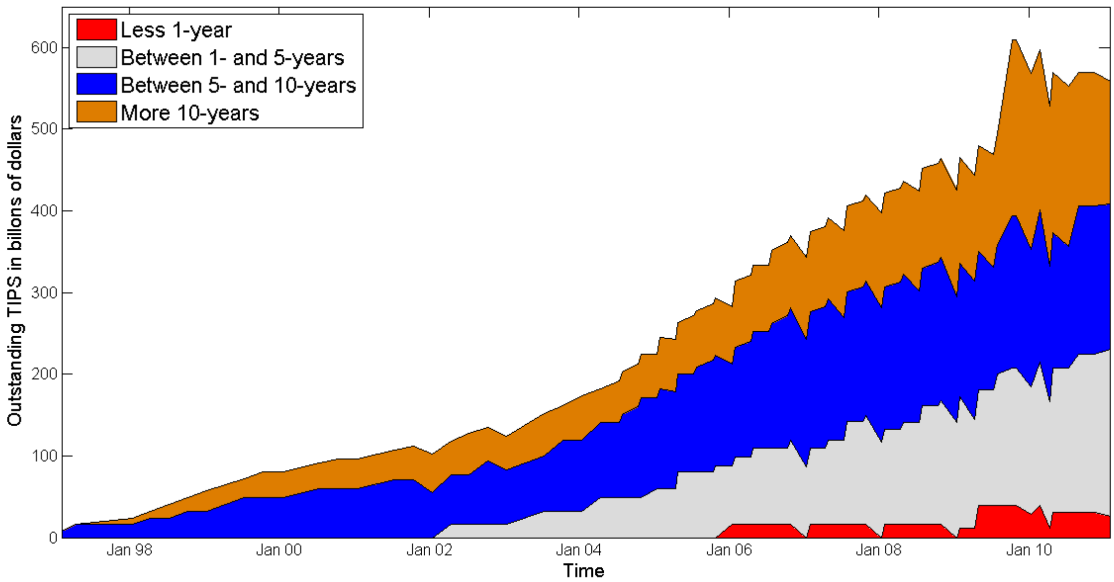
Equation (2.9) shows the case when the short-selling restriction for a risky asset is binding. If the expected return of asset i, corrected by risk, is lower than the benchmark asset, the investor sets his holdings in that particular asset to zero. Equation (2.10) restricts the investor from borrowing money at the nominal risk-free rate.

[10] Note that $\boldsymbol{\sigma}_0 = -\boldsymbol{\sigma}_{r\pi}$ since $r_0 = r_f^{\$} - \pi$ where $r_f^{\$}$ is the log nominal riskless rate and π is the log inflation rate.

2.4 Treasury Inflation-Protected Securities (TIPS)

The primary feature of TIPS is that their principal is indexed to the U.S. non-seasonally adjusted consumer price index for all urban consumers (CPI-U NSA). Then, if an investor holds TIPS until maturity, he will receive a known return in real terms for his investment. On the other hand, before maturity, TIPS' returns are uncertain both in real and nominal terms.

TIPS have been issued with maturities of 5, 10, 20, and 30 years. The 5-year TIPS were first issued toward the end of 1997, but TIPS with this maturity were discontinued until the end of 2004 when the Treasury started yearly issues. The 10-year class of TIPS is the only one which has been continuously issued. Initially, the 10-year TIPS were issued once a year, but from July 2003 they have been issued twice a year every year. Between 1998 and 2001 the Treasury issued three lots of 30-year TIPS (one lot every year) and were then discontinued until 2010 when they started issuing them again. Between 2005 and 2009 there were five yearly 20-year emissions.



We group available TIPS according to their term to maturity. The red area corresponds to TIPS with less than 1-year to maturity; the grey area corresponds to TIPS between 1-year and 5-years to maturity; the blue area corresponds to TIPS between 5-years and 10-years to maturity; and, the brown area corresponds to TIPS with more than 10-years to maturity.

Figure 2.1: Time series of TIPS available for investors

Figure 2.1 shows a time series of outstanding TIPS grouped according to their term to maturity. The lack of outstanding TIPS with maturity less than 1-year between 1997 and 2010 reflects the absence of a riskless asset in real terms for short-term investors. Although since 2005 there were more opportunities to find outstanding (off-the-run) TIPS with shorter maturities, they were primarily issued to provide a safe asset for investors with long investment horizon. Hence, while buy-and-hold long-term investors have been able to invest, but not borrow, at the real risk-free rate, short-term investors have had no access to a riskless asset in real terms and see TIPS as "risky" assets.

2.4.1 Data

We create a data set of nominal monthly returns for all TIPS issued before August 2009. A comparable nominal Treasury bond is selected as a benchmark for each TIPS, according to the issue and maturity date. The period of study spans from March 1997 to March 2010 which results in 157 monthly observations. The period encompasses an entire U.S. business cycle and two recessions.[11]

We group TIPS and nominal bonds according to their maturity: short-term (ST) notes (4 to 5 years), medium term (MT) notes (9 to 10 years) and long-term (LT) bonds (more than 10 years). See Table 2.1 and Table 2.2 for descriptive statistics. For TIPS indices, the reference security is the newest TIPS issuance within the group. For Nominal indices, we obtain a comparable nominal bond closest in maturity to the correspondent TIPS. Monthly TIPS returns are calculated using accrued interest, capital gains, and inflation adjustments. The real risk-free rate in the case of the long term investor corresponds to the yield of the corresponding TIPS in the index. In our optimization part we will use the MT notes that have an average maturity slightly below than 10 years.

Our approach to construct bond indices alleviates two potential concerns concerns that 1) off-the-run TIPS dominate the real bond index; and, 2) comparisons of real and nominal bonds is based on securities with vastly different maturities. The 10-year index constructed is an index that has an average maturity slightly smaller than 10 years. This index has a small roll down that takes places in between auctions as we drop the old TIPS and incorporate in the index the most recently issued. This roll down effect is not a very relevant for a part of of the curve (the segment between 9.5 and 10 years which is relatively flat. Moreover this solution outweighs the problems induced by the use of synthetic bonds or interpolated data. We believe our data choice is a better solution than constructing synthetic bonds or using constant maturity instruments. Both of these former measures are not transaction data and moreover parameter uncertainty in estimation and index construction might introduce small biases.

The rest of the data set is composed of alternative investment assets and a measure of inflation. For the different asset classes, we use a representative index: for equity we use the S&P500 index; for commodities we employ gold prices and the S&P GSCI index; and for real estate we use the S&P/Case-Shiller Home Price index. Inflation is measured as the log-difference of the CPI-U NSA for two consecutive months.

[11] http://www.nber.org/cycles/cyclesmain.html. All market price data are from Bloomberg.

Returns. Table 2.3 exhibits summary statistics of monthly excess log returns over the one-month T-bill for bond indices and the rest of the financial assets for three sample periods: Panel A: the entire period from March 1997 to March 2010; Panel B: the whole business cycle (from peak to peak) according to the Business Cycle Committee of the National Bureau of Economic Research, from March 2001 to December 2007; and, Panel C: the last two U.S. recessions, from March 2001 to November 2001 and from December 2007 to June 2009. Means, medians and standard deviations of excess log returns are annualized. Table 2.4 exposes the summary statistics that corresponds to the real log returns of risky assets.

Panel A in Table 2.3 shows that longer term TIPS experienced higher average returns than the shorter term maturities, suggesting the presence of a term premium in TIPS' returns. The long-term TIPS index had a mean monthly excess return[12] of 5.05% per year for the entire sample, while 5 and 10-year TIPS indices experienced a mean return of 2.82% and 3.10%, respectively. At the same time, TIPS have outperformed comparable nominal bonds for the complete period. Panel B corroborates the presence of a bond term premium in TIPS during the entire U.S. Business Cycle from March 2001 to December 2007. Panel B also shows that the outperformance of TIPS over nominal bonds is even higher during this sub-period.

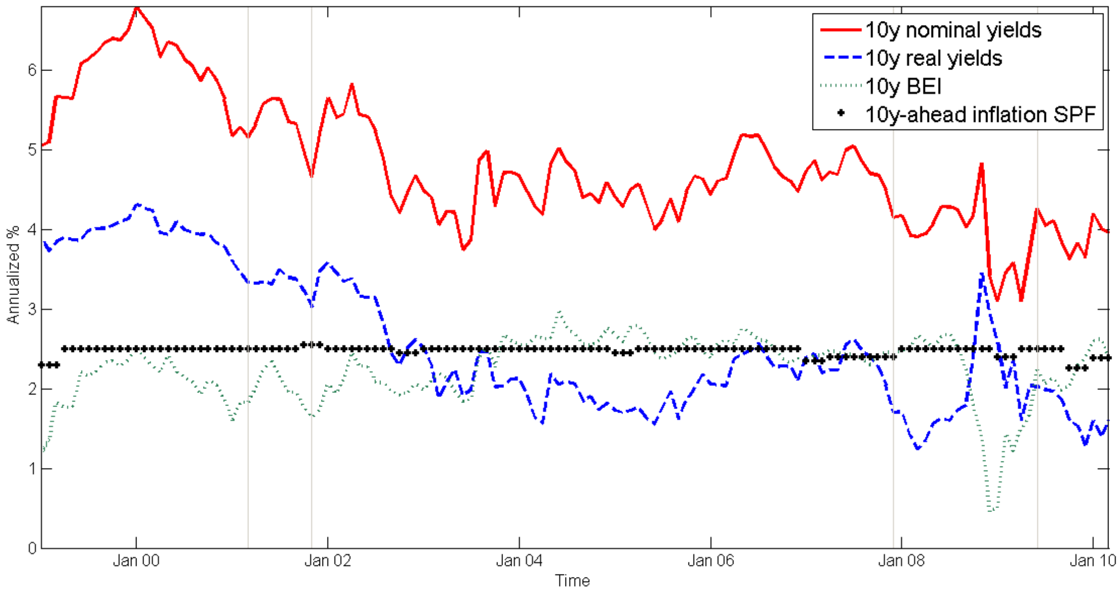
Interestingly, during the last two U.S. recessions (March 2001 to November 2001 and December 2007 to June 2009) nominal bonds outperformed TIPS' returns and short-term bonds showed higher realized excess log returns than long-term bonds, see Panel C in Table 2.3. The average inflation rates of the recession sub-sample was 1.20%, lower than the average observed for the entire business cycle period of 2.62%. During recessions one would expect short-term real interest rates to go down creating positive returns for bonds. Besides, in a framework of a stable Phillips curve inflation rates also will drop. Under this context, nominal bonds would be preferable during recessions than TIPS with similar maturity.

Real vs nominal yields. The introduction of TIPS allows long-term investors to invest in a risk-free asset that guarantees a real return. The long-term excess returns in real terms for the 10-year nominal bonds over the 10-year TIPS is -0.36% for the complete period. The general agreement about the negative premium is that it reflects the lower liquidity of TIPS relative to comparable nominal bonds.[13] A liquidity premium in TIPS over comparable nominal bonds makes sense when buyers of TIPS consider the chance to unwind the position before maturity. However, if all individuals in the economy were buy-and-hold long-term investors, the liquidity premium would be zero, and a negative long-term real excess return for nominal bonds would imply a negative inflation risk premium, which seems odd in a risk-averse world where individuals are not affected by

[12]Average returns are equal to mean log returns plus Jensen's correction, $\frac{\sigma^2}{2}$.

[13]See Sack and Elsasser (2004), D'Amico et al. (2010), Pflueger and Viceira (2010), among others. Fleckenstein et al. (2010) present the TIPS-Treasury bond puzzle by showing that TIPS are consistently undervalued relative to nominal U.S. Treasury bonds.

money illusion.



Nominal and real yields are 10-year constant maturity from GSW (2010). The 10-Year-Ahead Inflation Forecasts is from the Survey of Professional Forecasters.

Figure 2.2: Nominal and real yields

Volatility. The volatility of the excess log returns of TIPS is lower than the volatility of the returns of comparable nominal bonds when the entire period is considered. During the expansionary phase of the business cycle, TIPS' excess log returns also exhibit lower volatility than nominal bonds, except for long-term bonds.

As it was documented by Brière and Signori (2009), and Campbell et al. (2009), TIPS' volatility has increased relative to that of nominal bonds since 2003. Strikingly, during the financial crisis that started in 2008 the volatility of TIPS was equal or even higher than that of nominal bonds. TIPS' nominal returns are expected to have higher variance than the variance of the returns of nominal bonds when: the variance of realized inflation is higher than the variance of expectations about future inflation rates; or, the variance of any further premia included in TIPS is higher than the variance of any premia included in nominal bonds. Realized inflation exhibits low and almost constant volatility with an increase in the last part of 2008 due to a sharp decline in prices. At the same time, the large changes in break-even inflation rates driven by an increase in TIPS' real yields (Figure 2.2) in the same period suggest that both effects have pushed the variance of TIPS' return, measured in nominal terms, above the variance of the returns of nominal bonds (Figure 2.3).

Table 2.1: Sample of TIPS and comparable nominal Treasury bonds

We group TIPS and nominal bonds based on their maturity: short-term notes (4 to 5 years), medium term notes (9 to 10 years) and long-term bonds (more than 10 years). We obtain a comparable nominal bond by choosing the most recent on-the-run bond closest in maturity to our TIPS benchmark at the moment of the TIPS issuance.

	Date		Real			
	Issue	Maturity	Coupon %	Yield %	Price per $100	CUSIP
I) U.S. TIPS						
a) Short-term notes (4-year to 5-year)						
1.	15-Jul-97	15-Jul-02	3.625	3.744	99.462	9128273A8
2.	29-Apr-05	15-Apr-10	0.875	1.200	99.624	912828CZ1
3.	28-Apr-06	15-Apr-11	2.375	2.379	100.067	912828FB1
4.	30-Apr-07	15-Apr-12	2.000	2.114	99.731	912828GN4
5.	30-Apr-08	15-Apr-13	0.625	0.745	99.561	912828HW3
6.	30-Apr-09	15-Apr-14	1.250	1.278	100.113	912828KM1
		mean	1.792	1.910		
b) Notes (9-year to 10-year)						
1.	6-Feb-97	15-Jan-07	3.375	3.449	99.482	9128272M3
2.	15-Jan-98	15-Jan-08	3.625	3.730	99.130	9128273T7
3.	15-Jan-99	15-Jan-09	3.875	3.898	99.811	9128274Y5
4.	18-Jan-00	15-Jan-10	4.250	4.338	99.298	9128275W8
5.	16-Jan-01	15-Jan-11	3.500	3.522	99.818	9128276R8
6.	15-Jan-02	15-Jan-12	3.375	3.480	99.120	9128277J5
7.	15-Jul-02	15-Jul-12	3.000	3.099	99.154	912828AF7
8.	15-Jul-03	15-Jul-13	1.875	1.999	98.881	912828BD1
9.	15-Jan-04	15-Jan-14	2.000	2.019	99.829	912828BW9
10.	15-Jul-04	15-Jul-14	2.000	2.020	99.820	912828CP3
11.	18-Jan-05	15-Jan-15	1.625	1.725	99.091	912828DH0
12.	15-Jul-05	15-Jul-15	1.875	1.939	99.421	912828EA4
13.	17-Jan-06	15-Jan-16	2.000	2.025	99.723	912828ET3
14.	17-Jul-06	15-Jul-16	2.500	2.550	99.593	912828FL9
15.	16-Jan-07	15-Jan-17	2.375	2.449	99.342	912828GD6
16.	16-Jul-07	15-Jul-17	2.625	2.749	98.942	912828GX2
17.	15-Jan-08	15-Jan-18	1.625	1.655	99.725	912828HN3
18.	15-Jul-08	15-Jul-18	1.375	1.485	98.981	912828JE1
19.	15-Jan-09	15-Jan-19	2.125	2.245	98.931	912828JX9
20.	15-Jul-09	15-Jul-19	1.875	1.920	99.592	912828LA6
		mean	2.544	2.615		
c) Bonds (19-year to 20-year and 29-year to 30-year)						
1.	15-Apr-98	15-Apr-28	3.625	3.740	97.937	912810FD5
2.	15-Apr-99	15-Apr-29	3.875	3.899	99.578	912810FH6
3.	15-Oct-01	15-Apr-32	3.375	3.465	98.314	912810FQ6
4.	31-Jan-05	15-Jan-25	2.375	2.000	107.552	912810FR4
5.	31-Jan-06	15-Jan-26	2.000	2.039	98.949	912810FS2
6.	31-Jan-07	15-Jan-27	2.375	2.420	99.213	912810PS1
7.	31-Jan-08	15-Jan-28	1.750	1.807	99.351	912810PV4
8.	30-Jan-09	15-Jan-29	2.500	2.500	99.064	912810PZ5
		mean	2.734	2.734		

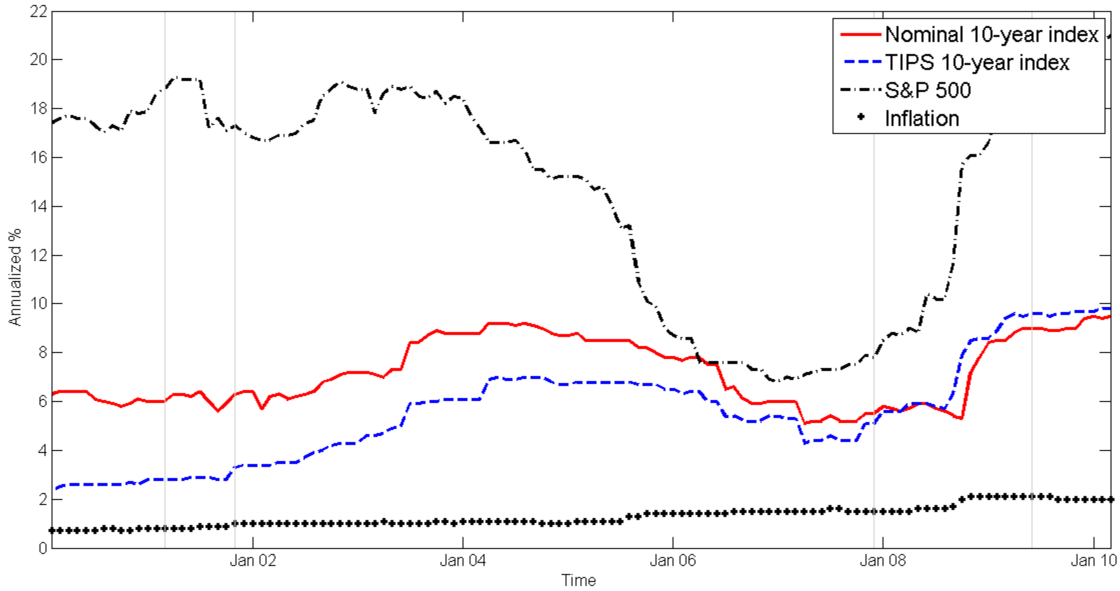
	Date		Nominal			
	Issue	Maturity	Coupon %	Yield %	Price per $100	CUSIP
II) U.S. Treasury Nominal Bonds						
a) Short-term notes (4-year to 5-year)						
1.	31-Jul-97	31-Jul-02	6.000	6.024	99.898	9128273C4
2.	15-Apr-05	15-Apr-10	4.000	4.046	99.794	912828DR8
3.	1-May-06	30-Apr-11	4.875	4.964	99.610	912828FD7
4.	30-Apr-07	30-Apr-12	4.500	4.579	99.651	912828GQ7
5.	30-Apr-08	30-Apr-13	3.125	3.159	99.844	912828HY9
6.	30-Apr-09	30-Apr-14	1.875	1.940	99.692	912828KN9
		mean	4.063	4.119		
b) Notes (9-year to 10-year)						
1.	18-Feb-97	15-Feb-07	6.250	6.374	99.092	9128272J0
2.	17-Feb-98	15-Feb-08	5.500	5.558	99.559	9128273X8
3.	16-Feb-99	15-Nov-08	4.750	4.913	98.735	9128274V1
4.	15-Feb-00	15-Feb-10	6.500	6.540	99.710	9128275Z1
5.	15-Feb-01	15-Feb-11	5.000	5.067	99.479	9128276T4
6.	15-Feb-02	15-Feb-12	4.875	4.880	99.961	9128277L0
7.	15-Aug-02	15-Aug-12	4.375	4.390	99.880	912828AJ9
8.	15-Aug-03	15-Aug-13	4.250	4.370	99.036	912828BH2
9.	17-Feb-04	15-Feb-14	4.000	4.060	99.511	912828CA6
10.	16-Aug-04	15-Aug-14	4.250	4.270	99.838	912828CT5
11.	15-Feb-05	15-Feb-15	4.000	4.049	99.600	912828DM9
12.	15-Aug-05	15-Aug-15	4.250	4.350	99.196	912828EE6
13.	15-Feb-06	15-Feb-16	4.500	4.540	99.681	912828EW6
14.	15-Aug-06	15-Aug-16	4.875	4.930	99.570	912828FQ8
15.	15-Feb-07	15-Feb-17	4.625	4.740	99.093	912828GH7
16.	15-Aug-07	15-Aug-17	4.750	4.855	99.176	912828HA1
17.	15-Feb-08	15-Feb-18	3.500	3.620	99.001	912828HR4
18.	15-Aug-08	15-Aug-18	4.000	4.075	99.389	912828JH4
19.	17-Feb-09	15-Feb-19	2.750	2.818	99.411	912828KD1
20.	17-Aug-09	15-Aug-19	3.625	3.734	99.097	912828LJ7
		mean	4.531	4.607		
c) Bonds (19-year to 20-year and 29-year to 30-year)						
1.	17-Aug-98	15-Aug-28	5.500	5.590	98.697	912810FE3
2.	16-Feb-99	15-Feb-29	5.250	5.298	99.282	912810FG8
3.	15-Aug-01	15-Feb-31	5.375	5.520	97.900	912810FP8
4.	15-Feb-95	15-Feb-25	7.620	7.650	99.708	912810ET1
5.	15-Feb-96	15-Feb-26	6.000	6.119	98.374	912810EW4
6.	18-Feb-97	15-Feb-27	6.625	6.640	99.804	912810EZ7
7.	17-Feb-98	15-Nov-27	6.125	5.822	104.238	912810FB9
8.	16-Feb-99	15-Feb-29	5.250	5.298	99.282	912810FG8
		mean	5.968	5.992		

Table 2.2: Nominal monthly log returns of TIPS and comparable nominal Treasury bonds

We group TIPS and nominal bonds based on their maturity: short-term notes (4 to 5 years), medium term notes (9 to 10 years) and long-term bonds (more than 10 years). We obtain a comparable nominal bond by choosing the most recent on-the-run bond closest in maturity to our TIPS benchmark at the moment of the TIPS issuance.

Date from/to	N	mean	med	monthly max	monthly min	std	skew	kurt	CUSIP
I) U.S. TIPS									
a) Short-term notes (4-year to 5-year)									
Aug-97/Feb-02	55	6.18	6.26	1.36	-0.22	1.34	0.09	2.32	9128273A8
Nov-04/Mar-10	65	3.41	3.54	2.40	-4.58	3.43	-1.67	10.67	912828CZ1
May-06/Mar-10	47	5.22	5.61	3.20	-6.24	4.85	-2.09	12.55	912828FB1
May-07/Mar-10	35	5.84	6.51	3.83	-6.06	5.89	-1.45	7.74	912828GN4
May-08/Mar-10	23	3.35	3.72	2.56	-4.09	5.43	-0.87	3.93	912828HW3
May-09/Mar-10	11	6.96	7.75	2.16	-1.06	3.67	-0.14	1.73	912828KM1
b) Notes (9-year to 10-year)									
Mar-97/Dec-06	118	5.73	5.24	2.48	-1.80	2.71	0.18	3.70	9128272M3
Feb-98/Dec-07	119	6.13	5.38	2.94	-2.34	3.08	0.06	4.20	9128273T7
Feb-99/Dec-08	119	6.32	6.57	3.11	-2.82	3.42	-0.18	4.66	9128274Y5
Feb-00/Dec-09	119	6.48	6.97	3.07	-3.76	4.03	-0.70	5.28	9128275W8
Feb-01/Mar-10	110	6.03	6.28	3.69	-6.24	4.93	-1.21	7.51	9128276R8
Feb-02/Mar-10	98	6.44	6.62	3.81	-6.67	5.44	-1.17	7.35	9128277J5
Aug-02/Mar-10	92	5.92	5.43	4.08	-6.41	5.83	-0.86	5.97	912828AF7
Aug-03/Mar-10	80	5.89	5.65	4.15	-6.28	5.91	-0.97	5.91	912828BD1
Feb-04/Mar-10	74	5.34	5.21	4.22	-8.75	6.59	-1.56	9.47	912828BW9
Aug-04/Mar-10	68	5.50	4.82	4.47	-8.25	6.62	-1.12	8.14	912828CP3
Feb-05/Mar-10	62	4.97	5.22	5.32	-10.31	7.52	-1.60	11.33	912828DH0
Aug-05/Mar-10	56	5.46	6.43	5.59	-11.14	8.27	-1.74	11.60	912828EA4
Feb-06/Mar-10	50	5.58	6.74	6.60	-10.62	8.88	-1.18	9.10	912828ET3
Aug-06/Mar-10	44	6.43	7.98	6.95	-10.49	9.42	-1.07	8.23	912828FL9
Feb-07/Mar-10	38	7.01	10.07	7.33	-10.05	10.15	-0.81	6.70	912828GD6
Aug-07/Mar-10	32	6.85	10.04	6.96	-9.21	10.48	-0.61	5.41	912828GX2
Feb-08/Mar-10	26	2.69	7.99	6.53	-9.02	10.62	-0.75	4.93	912828HN3
Aug-08/Mar-10	20	3.06	9.04	6.71	-8.12	11.58	-0.47	3.65	912828JE1
Feb-09/Mar-10	14	5.32	7.07	6.41	-3.41	8.77	0.61	3.36	912828JX9
Aug-09/Mar-10	8	6.33	14.76	2.70	-2.84	6.83	-0.62	2.00	912828LA6
c) Bonds (19-year to 20-year and 29-year to 30-year)									
May-98/Mar-10	143	7.40	7.03	10.16	-11.30	10.35	-0.54	5.56	912810FD5
May-99/Mar-10	131	7.94	7.32	10.09	-11.33	10.85	-0.56	5.23	912810FH6
Nov-01/Mar-10	101	7.00	6.83	13.74	-10.91	13.44	-0.17	4.44	912810FQ6
Aug-04/Mar-10	68	5.58	4.61	8.08	-11.96	10.59	-0.70	6.22	912810FR4
Feb-06/Mar-10	50	4.23	3.88	10.15	-11.46	12.42	-0.26	4.99	912810FS2
Feb-07/Mar-10	38	6.06	8.23	10.04	-11.86	13.76	-0.36	4.58	912810PS1
Feb-08/Mar-10	26	0.48	1.19	9.19	-11.76	16.17	-0.13	3.48	912810PV4
Feb-09/Mar-10	14	8.01	14.20	8.92	-5.95	13.14	0.24	3.02	912810PZ5

Date from/to	N	mean	med	monthly max	monthly min	std	skew	kurt	CUSIP
II) U.S. Treasury Nominal Bonds									
a) Short-term notes (4-year to 5-year)									
Aug-97/Feb-02	55	6.12	5.62	2.64	-1.54	2.38	0.24	5.06	9128273C4
May-05/Mar-10	59	3.88	3.06	2.41	-1.50	2.44	0.33	4.25	912828DR8
Jun-06/Mar-10	46	5.95	4.95	3.16	-1.63	3.04	0.61	4.53	912828FD7
May-07/Mar-10	35	6.55	4.60	3.44	-1.95	4.03	0.46	3.52	912828GQ7
May-08/Mar-10	23	5.01	6.56	4.48	-1.72	4.67	0.83	4.77	912828HY9
May-09/Mar-10	11	1.30	3.74	1.81	-2.37	4.42	-0.49	2.38	912828KN9
b) Notes (9-year to 10-year)									
Mar-97/Jan-07	119	6.07	4.52	4.35	-3.73	4.42	0.07	4.24	9128272J0
Mar-98/Jan-08	119	5.36	4.66	4.62	-4.28	4.62	0.02	4.99	9128273X8
Dec-98/Oct-08	119	4.67	4.53	3.51	-4.48	4.74	-0.44	4.63	9128274V1
Mar-00/Jan-10	119	5.80	5.06	3.71	-4.37	4.81	-0.32	4.54	9128275Z1
Mar-01/Mar-10	109	5.04	4.25	4.19	-5.33	5.50	-0.46	4.71	9128276T4
Mar-02/Mar-10	97	5.43	5.20	4.42	-5.96	5.79	-0.56	5.19	9128277L0
Sep-02/Mar-10	91	4.88	5.01	4.70	-6.44	5.99	-0.51	5.60	912828AJ9
Sep-03/Mar-10	79	5.55	7.54	4.57	-4.94	5.52	-0.22	4.17	912828BH2
Mar-04/Mar-10	73	5.06	7.39	4.75	-5.03	5.69	-0.30	4.00	912828CA6
Sep-04/Mar-10	67	5.35	7.47	4.86	-2.87	5.49	0.07	2.96	912828CT5
Mar-05/Mar-10	61	5.85	7.43	6.07	-2.89	6.03	0.33	3.55	912828DM9
Sep-05/Mar-10	55	5.34	6.08	7.60	-3.14	6.86	0.73	4.71	912828EE6
Mar-06/Mar-10	49	6.43	7.62	7.93	-3.46	7.17	0.70	4.88	912828EW6
Sep-06/Mar-10	43	6.96	7.86	7.88	-3.48	7.43	0.73	4.67	912828FQ8
Mar-07/Mar-10	37	6.86	6.64	7.73	-3.91	8.26	0.55	3.93	912828GH7
Sep-07/Mar-10	31	7.04	8.35	7.61	-4.02	8.81	0.53	3.63	912828HA1
Mar-08/Mar-10	25	3.64	4.69	7.72	-4.83	9.77	0.55	3.78	912828HR4
Sep-08/Mar-10	19	4.68	2.68	8.75	-4.91	11.38	0.74	3.75	912828JH4
Mar-09/Mar-10	13	-1.80	2.79	3.28	-4.60	8.11	-0.54	2.38	912828KD1
Sep-09/Mar-10	7	-1.63	1.78	2.34	-4.73	8.16	-1.03	3.15	912828LJ7
c) Bonds (19-year to 20-year and 29-year to 30-year)									
Sep-98/Mar-10	139	5.78	9.26	13.58	-10.72	11.39	-0.21	5.43	912810FE3
Mar-99/Mar-10	133	6.23	11.02	13.66	-11.22	11.58	-0.20	5.68	912810FG8
Mar-01/Mar-10	109	5.87	11.43	14.01	-12.02	12.76	-0.18	5.42	912810FP8
Aug-04/Mar-10	68	6.21	8.12	12.63	-8.63	10.63	0.52	6.35	912810ET1
Feb-06/Mar-10	50	5.33	6.70	12.74	-8.62	11.82	0.57	5.98	912810EW4
Feb-07/Mar-10	38	6.11	5.43	12.78	-8.85	12.87	0.55	5.53	912810EZ7
Feb-08/Mar-10	26	3.37	3.61	12.83	-9.80	15.46	0.50	4.60	912810FB9
Feb-09/Mar-10	14	-2.44	2.40	5.60	-6.33	11.36	-0.42	2.79	912810FG8



Standard deviation of returns are computed using a 3-year rolling window estimation.

Figure 2.3: Annualized standard deviation of returns

Correlations. Table 2.5 shows the correlation matrix of the set of financial assets for the entire period (Panel A) and for the full business cycle (Panel B). As expected, TIPS' excess log returns are highly correlated among them because they depend on the same factors; fluctuations in the real interest rate, changes in any premium contained in TIPS, and realized inflation. Besides, TIPS with adjacent maturities and with longer durations have higher correlations among them. Both panels in Table 2.5 exhibit the high correlation between TIPS and nominal bonds, particularly for bonds with longer maturities. The minimum correlation among Treasury bonds for the entire period is observed between the 5-year Nominal index and the LT TIPS Bond index, 0.58, while the maximum correlation of 0.79 is between the LT TIPS Bond index and the LT nominal Bond index.

An interesting point in both panels is the low correlation between realized inflation, measured by the CPI-U NSA, and TIPS' excess log returns. The longer the maturity of the TIPS, the lower the correlation between TIPS and realized inflation. In Panel A, the correlation of the TIPS 5-year index, 10-year index, and Bond index with inflation are 0.20, 0.16, and 0.08, respectively. If we consider data for the entire business cycle (Panel B), the correlation between TIPS' excess log returns and inflation is even lower. It is clear that realized inflation is not the predominant source of monthly TIPS returns which may suggest that they do not provide a good hedge against inflation over short periods. Commodities, rather than gold, seem to be a better instrument to protect investors against inflation.

Table 2.3: Excess monthly log returns.

A) Excess log returns of risky assets over the T-bill, $[r_i - r_0]$, for the complete period, March 1997 to March 2010. The mean, median, and standard deviation of monthly returns are annualized (in percentage terms). For each group of TIPS we construct an index return with the return of the on-the-run security. At any time within each TIPS group we obtain the most recent issued bond that will coincide with the newest issuance within the group. We obtain a comparable nominal bond by choosing the closest in maturity to our TIPS benchmark.

					monthly						
Financial Asset	N	**mean**	**median**	**avg(m, me)**	**max**	**min**	**std**	**skew**	**kurt**	**mean/std**	**med/std**
TIPS 5-year Index	152	2.75	1.25	2.00	3.45	-4.17	3.69	-0.248	5.252	0.745	0.339
TIPS 10-year Index	157	2.90	2.63	2.76	6.40	-8.20	6.27	-0.463	6.662	0.463	0.419
TIPS LT Bond Index	139	4.43	4.21	4.32	9.11	-11.84	11.12	-0.470	5.251	0.398	0.379
Nominal 5-year Index	152	2.18	1.75	1.97	4.44	-3.30	4.01	0.186	4.367	0.544	0.436
Nominal 10-year Index	157	2.55	2.43	2.49	8.72	-6.52	7.50	0.089	4.483	0.340	0.324
Nominal LT Bond Index	139	2.12	4.83	3.48	12.80	-11.19	11.27	-0.120	5.396	0.188	0.429
S&P 500 Index	157	1.33	10.25	5.79	9.13	-18.47	16.74	-0.828	4.182	0.079	0.612
Gold	157	5.41	-2.11	1.65	15.58	-12.89	13.89	0.252	4.415	0.389	-0.152
Commodity Index	157	-0.96	5.16	2.10	17.94	-33.21	24.72	-0.663	5.062	-0.039	0.209
Real Estate Index	155	2.02	4.49	3.25	2.18	-3.08	3.76	-0.919	3.626	0.537	1.194
Inflation (CPI-U NSA)	157	2.36	2.36	2.36	1.21	-1.93	1.36	-1.228	8.376	1.735	1.735

B) Excess log returns of risky assets over the T-bill, $[r_i - r_0]$, for the entire business cycle (peak to peak) according to the Business Cycle Committee of the National Bureau of Economic Research, March 2001 to December 2007.

					monthly						
Financial Asset	N	**mean**	**median**	**avg(m, me)**	**max**	**min**	**std**	**skew**	**kurt**	**mean/std**	**med/std**
TIPS 5-year Index	81	3.47	2.40	2.94	3.50	-2.80	3.76	0.044	3.911	0.923	0.638
TIPS 10-year Index	81	4.50	6.00	5.25	4.50	-4.90	5.86	-0.384	3.839	0.768	1.024
TIPS LT Bond Index	81	6.28	6.00	6.14	9.10	-10.30	10.83	-0.643	4.762	0.580	0.554
Nominal 5-year Index	81	2.19	2.40	2.30	3.10	-3.30	4.17	-0.141	3.742	0.525	0.576
Nominal 10-year Index	81	2.55	2.40	2.48	4.60	-6.50	7.39	-0.413	3.380	0.345	0.325
Nominal LT Bond Index	81	3.48	7.20	5.34	8.30	-11.20	10.76	-0.649	4.575	0.323	0.669
S&P 500 Index	81	1.42	9.60	5.51	8.30	-11.70	13.08	-0.553	3.773	0.109	0.734
Gold	81	13.53	7.20	10.37	9.70	-12.90	12.75	-0.375	4.554	1.061	0.565
Commodity Index	81	5.88	8.40	7.14	13.80	-15.70	21.75	-0.272	2.641	0.270	0.386
Real Estate Index	81	5.63	7.20	6.42	2.20	-2.50	3.28	-0.652	3.151	1.716	2.195
Inflation (CPI-U NSA)	81	2.62	2.40	5.02	1.20	-0.80	1.30	-0.157	2.862	2.015	1.846

C) Excess log returns of risky assets over the T-bill, $[r_i - r_0]$, for the two recessionary periods when TIPS were available, March 2001 to November 2001, and December 2007 to June 2009.

					monthly						
Financial Asset	N	**mean**	**median**	**avg(m, me)**	**max**	**min**	**std**	**skew**	**kurt**	**mean/std**	**med/std**
TIPS 5-year Index	28	1.62	1.20	1.41	2.60	-4.17	5.30	-0.581	3.809	0.306	0.226
TIPS 10-year Index	28	0.83	5.58	3.21	6.40	-8.20	10.18	-0.387	4.242	0.082	0.548
TIPS LT Bond Index	28	0.59	-4.20	-1.81	8.96	-11.84	15.66	-0.143	3.674	0.038	-0.268
Nominal 5-year Index	28	3.89	3.78	3.84	4.44	-2.14	4.67	0.840	4.758	0.833	0.809
Nominal 10-year Index	28	3.57	-0.36	1.61	8.72	-4.92	10.39	0.685	3.758	0.344	-0.035
Nominal LT Bond Index	28	3.28	-0.60	1.34	12.80	-9.81	16.35	0.513	3.814	0.201	-0.037
S&P 500 Index	28	-24.16	-12.36	-18.26	9.13	-18.47	23.18	-0.276	2.670	-1.042	-0.533
Gold	28	6.31	-4.08	1.12	9.87	-11.59	16.74	-0.120	3.119	0.377	-0.244
Commodity Index	28	-36.86	-27.84	-32.35	17.94	-33.21	35.55	-0.580	4.121	-1.037	-0.783
Real Estate Index	28	-11.71	-14.10	-12.91	1.41	-3.08	4.79	0.093	1.481	-2.445	-2.944
Inflation (CPI-U NSA)	28	1.19	3.24	4.43	1.00	-1.93	2.27	-1.234	4.670	0.524	1.427

Table 2.4: Real monthly log returns

A) Real monthly log returns for the complete period, March 1997 to March 2010. The mean, median, and standard deviation of monthly returns are annualized (in percentage terms). For each group of TIPS we construct an index return with the return of the on-the-run security. At any time within each TIPS group we obtain the most recent issued bond that will coincide with the newest issuance within the group. We obtain a comparable nominal bond by choosing the closest in maturity to our TIPS benchmark.

					monthly						
Financial Asset	*N*	**mean**	**median**	**avg(m, me)**	**max**	**min**	**std**	**skew**	**kurt**	**mean/std**	**med/std**
TIPS 5-year Index	152	3.68	3.07	3.38	3.90	-3.26	3.63	-0.040	4.922	1.014	0.847
TIPS 10-year Index	157	3.93	3.42	3.68	6.39	-6.18	6.17	-0.170	5.599	0.638	0.554
TIPS LT Bond Index	139	5.08	5.46	5.27	9.10	-10.81	11.07	-0.392	4.844	0.459	0.493
Nominal 5-year Index	152	3.11	3.12	3.12	5.52	-3.80	4.55	0.143	5.037	0.685	0.686
Nominal 10-year Index	157	3.59	5.71	4.65	9.79	-6.82	7.84	0.046	4.956	0.458	0.729
Nominal LT Bond Index	139	2.77	5.91	4.34	13.87	-11.49	11.51	-0.113	5.878	0.240	0.514
S&P 500 Index	157	2.37	7.75	5.06	9.27	-16.45	16.61	-0.712	3.769	0.142	0.467
Gold	157	6.45	0.27	3.36	15.83	-12.68	13.61	0.303	4.608	0.474	0.020
Commodity Index	157	0.07	3.77	1.92	17.10	-31.19	23.83	-0.634	4.821	0.003	0.158
Real Estate Index	155	3.11	5.59	4.35	2.42	-3.69	3.89	-1.175	4.459	0.801	1.439

B) Real monthly log returns for the entire business cycle (peak to peak) according to the Business Cycle Committee of the National Bureau of Economic Research, March 2001 to December 2007.

					monthly						
Financial Asset	*N*	**mean**	**median**	**avg(m, me)**	**max**	**min**	**std**	**skew**	**kurt**	**mean/std**	**med/std**
TIPS 5-year Index	81	3.68	3.06	3.37	3.90	-3.26	3.83	-0.040	4.922	0.960	0.799
TIPS 10-year Index	81	4.54	3.84	4.19	6.39	-6.18	5.95	-0.170	5.599	0.764	0.646
TIPS LT Bond Index	81	6.41	6.59	6.50	9.10	-10.81	10.91	-0.392	4.844	0.587	0.604
Nominal 5-year Index	81	2.47	2.44	2.45	5.52	-3.80	4.50	0.143	5.037	0.549	0.542
Nominal 10-year Index	81	2.82	6.23	4.52	9.79	-6.82	7.58	0.046	4.956	0.372	0.821
Nominal LT Bond Index	81	3.96	9.67	6.81	13.87	-11.49	10.87	-0.113	5.878	0.364	0.890
S&P 500 Index	81	2.62	7.27	4.95	9.27	-16.45	12.98	-0.712	3.769	0.202	0.560
Gold	81	15.58	11.73	13.65	15.83	-12.68	13.03	0.303	4.608	1.196	0.900
Commodity Index	81	7.70	13.21	10.46	17.10	-31.19	20.87	-0.634	4.821	0.369	0.633
Real Estate Index	81	5.41	7.37	6.39	2.42	-3.69	3.45	-1.175	4.459	1.569	2.137

C) Real monthly log returns for the two recessionary periods when TIPS were available, March 2001 to November 2001, and December 2007 to June 2009.

					monthly						
Financial Asset	*N*	**mean**	**median**	**avg(m, me)**	**max**	**min**	**std**	**skew**	**kurt**	**mean/std**	**med/std**
TIPS 5-year Index	28	2.65	3.63	3.14	3.90	-3.26	4.74	-0.040	4.922	0.559	0.766
TIPS 10-year Index	28	1.84	0.87	1.35	6.39	-6.18	9.57	-0.170	5.599	0.193	0.090
TIPS LT Bond Index	28	1.60	-1.90	-0.15	9.10	-10.81	15.13	-0.392	4.844	0.106	-0.126
Nominal 5-year Index	28	4.90	6.21	5.56	5.52	-3.80	6.16	0.143	5.037	0.796	1.008
Nominal 10-year Index	28	4.58	5.98	5.28	9.79	-6.82	11.22	0.046	4.956	0.408	0.533
Nominal LT Bond Index	28	4.29	6.43	5.36	13.87	-11.49	16.88	-0.113	5.878	0.254	0.381
S&P 500 Index	28	-23.15	-12.16	-17.66	9.27	-16.45	22.21	-0.712	3.769	-1.042	-0.547
Gold	28	7.32	-2.17	2.58	15.83	-12.68	16.14	0.303	4.608	0.454	-0.134
Commodity Index	28	-35.84	-29.07	-32.45	17.10	-31.19	33.75	-0.634	4.821	-1.062	-0.861
Real Estate Index	28	-10.70	-11.22	-10.96	2.42	-3.69	5.42	-1.175	4.459	-1.973	-2.068

Table 2.5: Correlations matrix of excess log returns

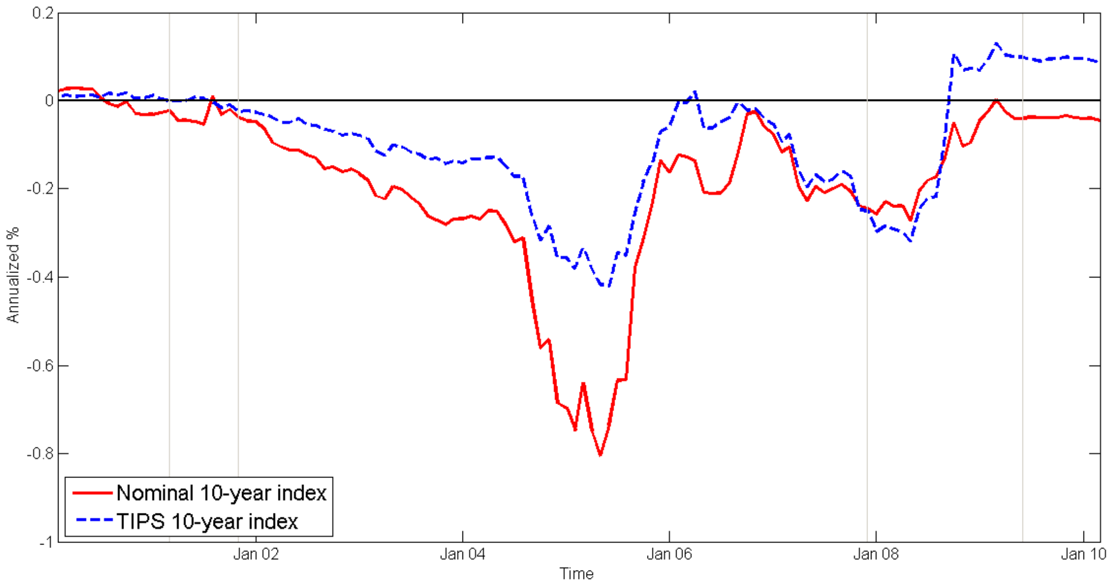
A) Correlations matrix of excess log returns of risky assets over the T-bill, $[\mathbf{r} - r_0\mathbf{1}]$, for the period March 1997 to March 2010.

			TIPS			Nominal						
		Inflation (1)	**5-year** (2)	**10-year** (3)	**LT Bond** (4)	**5-year** (5)	**10-year** (6)	**LT Bond** (7)	**S&P 500** (8)	**Gold** (9)	**S&P GSCI** (10)	**S&P CS10** (11)
(1)	**Inflation (CPI-NSA)**	1.000	0.202	0.163	0.088	-0.255	-0.152	-0.115	0.164	0.177	0.694	0.066
(2)	**TIPS 5-year Index**	0.202	1.000	0.920	0.813	0.671	0.671	0.617	-0.052	0.172	0.298	0.005
(3)	**TIPS 10-year Index**	0.163	0.920	1.000	0.896	0.647	0.738	0.706	0.020	0.115	0.244	0.021
(4)	**TIPS LT Bond Index**	0.088	0.813	0.896	1.000	0.587	0.741	0.787	0.051	0.040	0.203	0.058
(5)	**Nominal 5-year Index**	-0.255	0.671	0.647	0.587	1.000	0.873	0.747	-0.288	0.030	-0.139	-0.136
(6)	**Nominal 10-year Index**	-0.152	0.671	0.738	0.741	0.873	1.000	0.943	-0.213	-0.008	-0.053	-0.076
(7)	**Nominal LT Bond Index**	-0.115	0.617	0.706	0.787	0.747	0.943	1.000	-0.125	-0.018	-0.007	-0.035
(8)	**S&P 500 Index**	0.164	-0.052	0.020	0.051	-0.288	-0.213	-0.125	1.000	0.033	0.209	0.181
(9)	**Gold**	0.177	0.172	0.115	0.040	0.030	-0.008	-0.018	0.033	1.000	0.255	-0.055
(10)	**Commodity Index**	0.694	0.298	0.244	0.203	-0.139	-0.053	-0.007	0.209	0.255	1.000	0.241
(11)	**Real Estate Index**	0.066	0.005	0.021	0.058	-0.136	-0.076	-0.035	0.181	-0.055	0.241	1.000

B) Correlations matrix of excess log returns of risky assets over the T-bill, $[\mathbf{r} - r_0\mathbf{1}]$. Sample data of an entire business cycle (peak from previous peak) from March 2001 to December 2007, according to the Business Cycle Committee of the National Bureau of Economic Research.

			TIPS			Nominal						
		Inflation (1)	**5-year** (2)	**10-year** (3)	**LT Bond** (4)	**5-year** (5)	**10-year** (6)	**LT Bond** (7)	**S&P 500** (8)	**Gold** (9)	**S&P GSCI** (10)	**S&P CS10** (11)
(1)	**Inflation (CPI-NSA)**	1.000	0.054	0.028	-0.027	-0.120	-0.081	-0.071	-0.108	0.150	0.629	-0.001
(2)	**TIPS 5-year Index**	0.054	1.000	0.936	0.813	0.835	0.808	0.720	-0.379	-0.024	0.235	-0.024
(3)	**TIPS 10-year Index**	0.028	0.936	1.000	0.909	0.795	0.869	0.827	-0.338	-0.056	0.219	0.005
(4)	**TIPS Bond Index**	-0.027	0.813	0.909	1.000	0.696	0.810	0.852	-0.191	-0.048	0.216	0.014
(5)	**Nominal 5-year Index**	-0.120	0.835	0.795	0.696	1.000	0.877	0.754	-0.407	-0.061	-0.023	-0.059
(6)	**Nominal 10-year Index**	-0.081	0.808	0.869	0.810	0.877	1.000	0.944	-0.469	-0.114	0.015	-0.025
(7)	**Nominal Bond Index**	-0.071	0.720	0.827	0.852	0.754	0.944	1.000	-0.328	-0.096	0.054	-0.016
(8)	**S&P 500 Index**	-0.108	-0.379	-0.338	-0.191	-0.407	-0.469	-0.328	1.000	0.223	-0.042	-0.070
(10)	**Gold**	0.150	-0.024	-0.056	-0.048	-0.061	-0.114	-0.096	0.223	1.000	0.257	-0.069
(11)	**Commodity Index**	0.629	0.235	0.219	0.216	-0.023	0.015	0.054	-0.042	0.257	1.000	0.109
(12)	**Real Estate Index**	-0.001	-0.024	0.005	0.014	-0.059	-0.025	-0.016	-0.070	-0.069	0.109	1.000

Table 2.5 shows that the correlation between the stock market and nominal bonds is lower than the correlation between the stock market and TIPS. Figure 2.4 plots a 3-year rolling beta of the TIPS and nominal 10-year indices.[14] Nominal bonds seem to offer a better hedge against the equity market than TIPS, at least over short horizons. Under the CAPM framework this would imply a positive risk premium contained in TIPS relative to nominal bonds.



The beta is calculated as the covariance between the bonds' returns and the S&P 500 returns divided by the variance of the S&P 500 returns.

Figure 2.4: Conditional beta 3-year rolling window

2.5 Measuring the Benefits of TIPS

In this section we compute and analyze the benefits that TIPS provide to investors. Our empirical study focuses on two aspects. First, we consider different features about investors' preferences: investment horizon, and risk aversion. Second, we study how TIPS perform as a marginal security in the presence of other investment opportunities: nominal bonds, equity, commodities, gold, and real estate.[15]

To illustrate the marginal benefit of TIPS we employ three different measures. First, we compute the Risk-Adjusted Returns Benefits ($RARB$) which is the difference between

[14] The beta is calculated as the covariance between the bonds returns and the S&P 500 returns divided by the variance of the S&P 500 returns.

[15] Here we make the assumption that there is a real estate index that investors can purchase and liquidate with no transaction costs.

the return of the optimal portfolio which includes TIPS and the risk-adjusted expected return of the optimal portfolio, excluding TIPS,

$$RARB = \mathrm{E}_t\left[r_{p,t+1}^{(tips)}\right] - \frac{\sigma_{r_p^{(tips)}}}{\sigma_{r_p}}\mathrm{E}_t\left[r_{p,t+1}\right]. \tag{2.11}$$

This risk-adjusted measure has the significant advantage of being in basis points of returns which makes it simple to interpret, see Modigliani and Modigliani (1997). Additionally, we compute the ratio of the reward per unit of risk under the optimal allocation with TIPS and the reward per unit of risk under the optimal allocation without TIPS

$$RB = \frac{\mathrm{E}_t\left[r_{p,t+1}^{(tips)}\right]/\sigma_{r_p^{(tips)}}}{\mathrm{E}_t\left[r_{p,t+1}\right]/\sigma_{r_p}}. \tag{2.12}$$

The advantage of computing RB is that it helps to complement the risk-adjusted measure since it provides a relative measure of the gains in the reward per unit of risk. Finally, we compute the increment of the certainty equivalent under the optimal allocation with and without TIPS

$$BCE = r_p^{CE(tips)} - r_p^{CE}, \tag{2.13}$$

where $r_p^{CE} = \mathrm{E}_t\left[r_{p,t+1}\right] + (1-\gamma)\frac{\sigma_{r_p}^2}{2}$ is the certain real log return that the investor would accept rather than taking a chance on a higher, but uncertain, real log return.[16]

As discussed above, when the horizon over which investors hold assets does not match the maturity of any of the outstanding TIPS, these inflation protected bonds are risky and are considered by the investor as any other asset with uncertain payoff. On the other hand, if the horizon of investors and the maturity of TIPS coincide, then investors who maximize real wealth consider TIPS as the risk-free asset. Since most of the outstanding TIPS have maturities of more than one year, we regard those who have an investment horizon of one month and cannot find TIPS with this maturity short-term investors in our empirical study. Likewise, we regard those with a horizon of 10 years as buy-and-hold long-term investors because 10-year TIPS are the only ones who have been continuously issued since 1997.[17] Furthermore, throughout our analysis we assume that investors make decisions in real terms; that is, investors do not suffer from money illusion.

[16]Note that r_p^{CE} is obtained by replacing (2.2) into (2.1), equating $E_t[U(.)] = U(.)$ and taking into account that $(1 + R_{p,t+1}) \sim \text{log-}\mathcal{N}(\mu_{r_p}, \sigma_{r_p}^2)$ which results in

$$\frac{W_t^{1-\gamma}}{1-\gamma}e^{(1-\gamma)\mu_{r_p} + \frac{(1-\gamma)^2}{2}\sigma_{r_p}^2} = \frac{W_t^{1-\gamma}}{1-\gamma}e^{(1-\gamma)r_p^{CE}}, \quad \text{since} \quad \mathrm{E}_t\left[(1+R_{p,t+1})^{1-\gamma}\right] = e^{(1-\gamma)\mu_{r_p} + \frac{(1-\gamma)^2}{2}\sigma_{r_p}^2}.$$

[17]Fleming and Krishnan (2009) provide evidence that trading activity in TIPS is concentrated in 10-year notes with over 71.6 % of transactions taking places in this maturity segment.

Table 2.6: Inputs for the model

A) Inputs for the case of short-term investors correspond to excess log returns of risky assets over the T-bill, $[\mathbf{r} - r_0\mathbf{1}]$, for the period March 1997 to March 2010. The mean, median, and standard deviation are annualized (in percentage terms). T-bills are expressed in real terms.

						correlation matrix					
	Financial Asset	**mean**	**median**	**avg(m, me)**	**std**	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**
(1)	Inflation	2.36	2.36	2.36	1.36	0.164	-0.152	0.163	0.177	0.694	0.066
(2)	Stocks	1.33	10.25	5.79	16.74	1.000	-0.213	0.020	0.033	0.209	0.181
(3)	10-year nominal bonds	2.55	2.43	2.49	7.50	-0.213	1.000	0.738	-0.008	-0.053	-0.076
(4)	10-year TIPS	2.90	2.63	2.76	6.27	0.020	0.738	1.000	0.115	0.244	0.021
(5)	Gold	5.41	-2.11	1.65	13.89	0.033	-0.008	0.115	1.000	0.255	-0.055
(6)	Commodities	-0.96	5.16	2.10	24.72	0.209	-0.053	0.244	0.255	1.000	0.241
(7)	Real estate	2.02	4.49	3.25	3.76	0.181	-0.076	0.021	-0.055	0.241	1.000
(8)	T-bills	1.04	1.58	1.31	1.36	-0.164	0.152	-0.163	-0.177	-0.694	-0.066

B) Inputs for the case of long-term investors correspond to real monthly log returns of risky assets for the complete period. The mean, median, and standard deviation are annualized (in percentage terms). The real returns of the 10-year TIPS for the case of long-term investors refer to the bonds' yields while the real returns of the 10-year nominal bonds refer to their yield to maturity minus expected inflation.

						correlation matrix				
	Financial Asset	**mean**	**median**	**avg(m, me)**	**std**	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
(1)	Inflation	2.36	2.36	2.36	1.36	0.080	0.081	0.664	-0.275	-1.000
(2)	Stocks	2.37	7.75	5.06	16.61	1.000	0.007	0.154	0.143	-0.080
(3)	Gold	6.45	0.27	3.36	13.61	0.007	1.000	0.191	-0.106	-0.081
(4)	Commodities	0.07	3.77	1.92	23.83	0.154	0.191	1.000	0.014	-0.664
(5)	Real estate	3.11	5.59	4.35	3.89	0.143	-0.106	0.014	1.000	0.275
(6)	10-year nominal bonds	2.24	2.11	2.18	1.36	-0.080	-0.081	-0.664	0.275	1.000
(7)	10-year TIPS	2.61	2.35	2.48						

Specifically, we use the one-period portfolio choice framework in real terms, developed in Section 2.3, to measure the benefits that TIPS provide to both short- and long-term investors. We consider a range of values for risk aversion and measure the marginal benefit of TIPS in the presence of different asset classes. In particular, for short-term investors we consider the combination of stocks, nominal bonds, commodities, real estate, and T-bills. Here T-bills represent the short-term riskless asset in nominal terms so the riskiness in real terms of this security is given by the variation in inflation. In this case, the problem solved by investors is given by (2.7) and the optimal allocation is given in (2.8)-(2.10). Inputs for this allocation problem are exhibited in Panel A of Table 2.6. For long-term investors we consider the same scenarios, but we exclude T-bills and instead we use TIPS to be the riskless asset in real terms. In this case, we employ equations (2.3)-(2.6) with the inputs presented in Panel B of Table 2.6 to compute the optimal weights and to measure the benefits that TIPS provide to buy-and-hold long-term investors who are not affected by money illusion.

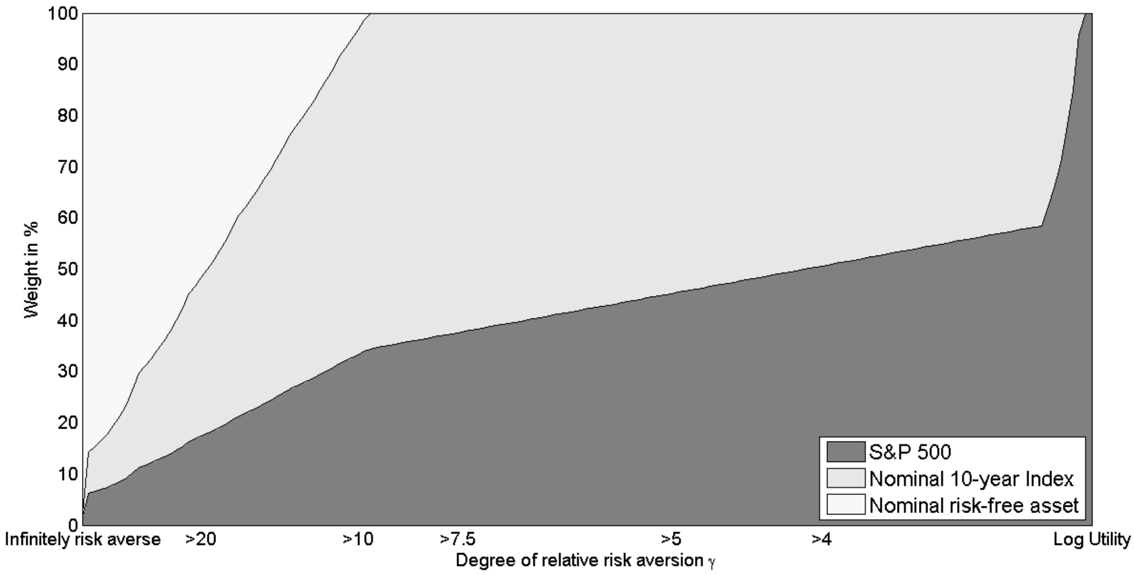
The existence of large outliers in returns for some assets during the recessionary periods (particularly during the 2008 financial crisis), together with the short time span available for TIPS (only two recessions and a full peak to peak expansion covered within the data) might misrepresent the ex-post sample mean as a true estimate of expected returns. This shortcoming has been addressed in the literature in different ways. Roll (2004) advises "that one should not blindly accept historical estimates is a forward looking portfolio allocation problem" and considers historical mean returns as unreliable estimates of future expected returns. He uses an arbitrarily chosen expected return figure (he gives stocks a 4% premium over the one year nominal bond yield) and estimates variances and covariances using a subset of his most recent sample data. Kothari and Shanken (2004) also argue that it is better not to use the ex-post sample mean because it might not reflect market participants' view of expected returns. Moreover, they argue that portfolio weights could be quite sensitive to relatively small changes in the expected return inputs. On the other hand, they consider that historical estimates of variances and covariances can be more reliable than sample means. Similarly, Brière and Signori (2009) use three baseline excess return scenarios as measures of expected returns and use historical data to estimate the conditional variances and covariances when considering the TIPS portfolio allocation problem. Finally, Hughson et al. (2006) also criticize the sample mean as an estimate of expected returns. The bootstrap simulations results in their article suggest that a better measure of expected returns and a better measure of location lies between the mean and the median.

In our study we employ the average of the ex-post sample mean and the ex-post sample median as the market's expected return for all variables employed in our model. And finally, we proceed as Kothari and Shanken (2004), Brière and Signori (2009) and employ historical data to estimate variances and covariances of all variables employed in our model. As a cross validation exercise we have also rerun the portfolio allocation problems using the sample mean and the median as the expected return. Our findings are qualitatively the same as those presented here.[18]

2.5.1 Short-term investors

We analyze the problem of short-term investors who maximize real wealth and are not able to invest in a riskless asset in real terms. In the absence of a riskless asset in real terms, the only way in which short-term investors can deal with uncertainty about inflation is through the covariances between the nominal returns of risky assets and inflation. For example, infinitely risk averse investors who are not affected by money illusion and are not allowed to short-sell assets, will invest a fraction of their wealth in risky assets which are positively correlated with inflation in order to reduce the portfolio variance in real terms.

[18]In the interest of space we do not include the results in the chapter but these are available from the authors upon request.



Optimal portfolio choice when a short-term investor is able to allocate his wealth into stocks, nominal bonds, and cash.

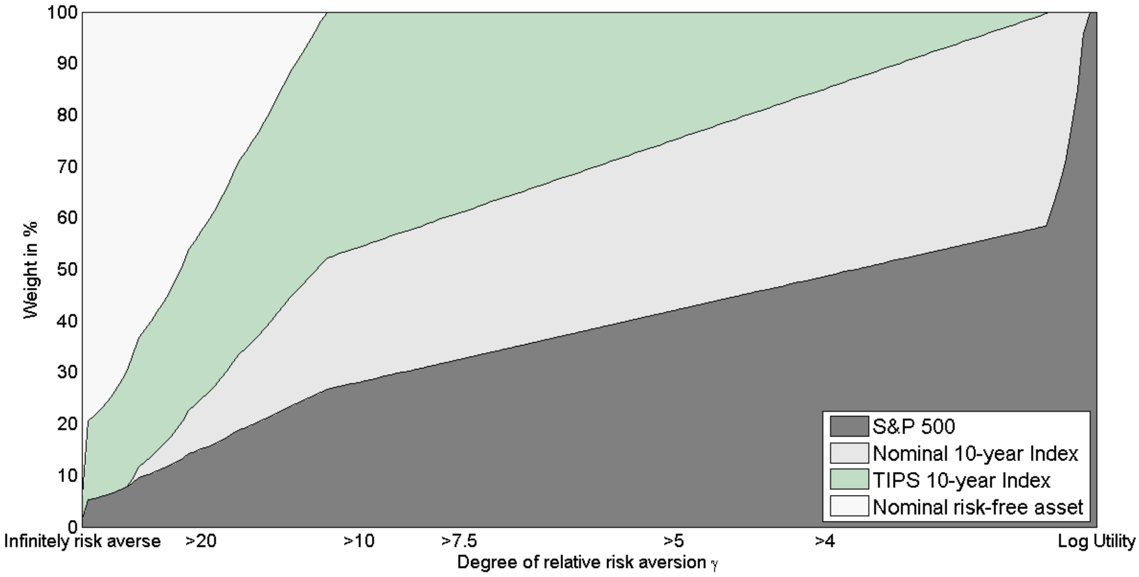
Figure 2.5: Optimal portfolio weights for a short-term investor without TIPS

Stocks, nominal bonds, and T-bills. To compare the marginal benefits of TIPS we use the case where short-term investors allocate their wealth to stocks, nominal bonds, and T-bills as a benchmark. Figure 2.5 shows the optimal weights, in percentage terms, of total invested wealth for different levels of relative risk aversion. The figure shows that for high (low) levels of risk aversion, the investor relies heavily on T-bills (stocks and 10-year nominal bonds) and that very little is invested in nominal 10-year bonds and stocks (T-bills). The absence of a riskless real asset, together with short-selling constraints, are able to explain "the asset allocation puzzle" stated by Canner et al. (1997); that is, aggressive investors hold a lower ratio of bonds to stocks compared with conservative investors.[19]

Stocks, 10-year nominal bonds, T-bills and 10-year TIPS. Figure 2.6 exhibits the optimal weights when TIPS are introduced. For all levels of relative risk aversion, except $\gamma \leq 3$, a significant part of investors' wealth is devoted to TIPS. Panel i in Table 2.7 shows that the optimal wealth devoted to TIPS varies from 3.5% for infinitely risk averse investors to 46.1% for investors with a degree of risk aversion of $\gamma = 10$. In this Table we show the optimal weight of each asset when TIPS are available and in parenthesis the optimal weight when TIPS are not available. Panel i shows that TIPS seem to be a reasonable, but not perfect, substitute for nominal bonds with the same maturity. By looking at the difference between the optimal weights with and without TIPS we show the

[19]Canner et al. (1997) document that financial advisors recommend that more risk averse investors should hold a higher ratio of bonds to stocks, which is inconsistent with the mutual-fund separation theorem. Under certain assumptions the theorem predicts that all investors should hold risky assets in the same proportion.

crowding out effects of TIPS. For instance for $\gamma = 5$ the optimal weight devoted to long-term nominal bonds is almost halved after introducing TIPS. Therefore, both classes of long-term bonds are included in the optimal portfolio of short-term investors. Extremely risk averse investors who maximize real wealth, invest a small fraction of their wealth into stocks (1.3%) and TIPS (3.5%) to hedge the inflation risk of T-bills.



Optimal portfolio choice when a short-term investor is able to allocate his wealth into stocks, nominal bonds, TIPS and cash.

Figure 2.6: Optimal portfolio weights for a short-term investor with TIPS

In order to analyze the benefits of TIPS to investors, we calculate (2.11) and the two relative measures, RB and BCE. Panel i in Table 2.7 reports the benefits of introducing TIPS for short-term investors over the period 1997 to 2010. For instance, while investors characterized by $\gamma = 10$ obtain a risk-adjusted returns benefit of $28bp$, extremely risk averse investors get almost $12bp$ by the introduction of TIPS which represents an increase in the portfolio's reward per unit of risk of 5.6% and 8.6% respectively. The increment in the certainty equivalent for investors characterized by $\gamma = 10$ is 0.2%. The positive correlation of TIPS' returns and inflation makes TIPS a useful instrument to reduce the inflation risk. On the other extreme, log-utility investors do not obtain any benefits from the introduction of TIPS.

If we restrict our data to cover the entire business cycle from March 2001 to December 2007, the benefits to investors from the introduction of TIPS are greater than those for the whole sample. Also, long-term nominal bonds are totally crowded out by long-term TIPS in the optimal portfolio of short-term investors. On the other hand, during recessionary periods (March 2001 to November 2001 and from December 2007 to June 2009) when average inflation rates are lower, long-term nominal bonds outperform long-term TIPS, with the latter providing no benefits to short-term investors.[20]

Stocks, nominal bonds, commodities, real estate, T-bills and TIPS. When commodities are available, the improvement decreases because commodities are a better hedge than TIPS against inflation.[21] For example, for levels of risk aversion around $\gamma = 10$ we see that the introduction of TIPS produces RARB of $17bp$ ($RB = 3.3\%, BCE = 0.08\%$), see Panel iii in Table 2.7. Interestingly, the fraction of wealth invested in commodities is low for all levels of risk aversion underlying the fact that although commodities benefit from unexpected spikes of inflation they do not provide a reliable hedge in real terms due to the high volatility of their returns.

Panels ii-v in Table 2.7 show that when short-term investors have a wider investment opportunity set that might include gold, commodities or real estate, the relative benefits from the introduction of TIPS diminish, but still remain part of their optimal allocation. The positive correlation of TIPS' returns with inflation makes TIPS desirable for highly risk averse investors since it enables them to reduce the portfolio variance in real terms. Lastly, when all assets are considered only risk averse investors with $\gamma = 20$ (approximately) do obtain benefits from TIPS. For instance, Panel v in Table 2.7 shows that investors with a degree of risk aversion of $\gamma = 20$ devote more than 9% of their wealth to TIPS. In this case, there is a relative benefit to investors of 1.1% measured in terms of the increment of the reward per unit of risk.

2.5.2 Buy-and-hold long-term investors

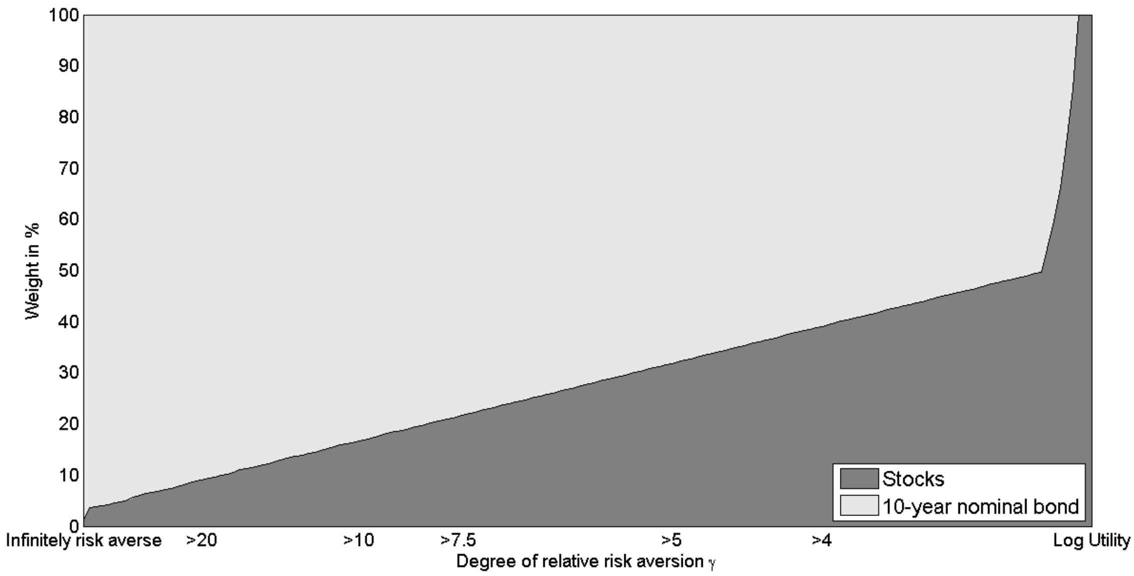
The introduction of TIPS provides buy-and-hold long-term investors with a risk-free asset in real terms. We assume that investors are neither allowed to borrow at the riskless rate in real terms, nor allowed to short-sell risky assets.

Stocks and 10-year nominal bonds. Figure 2.7 shows the optimal allocation weights when long-term investors can only purchase stocks and long-term nominal bonds with maturity of 10 years which is the same as the time horizon over which the buy-and-hold investor solves the optimal allocation problem. Infinitely risk averse investors allocate a very small fraction of their wealth to stocks, instead investing all in the long-term nominal bonds. Whereas for investors with log-utility function all wealth is placed in stocks.

Stocks, 10-year nominal bonds, and 10-year TIPS. Figure 2.8 exhibits the optimal allocation of the investor's wealth after the introduction of TIPS. The introduction of TIPS allows long-term investors to buy an asset that guarantees a real return. The long-term excess returns in real terms for stock and nominal long-term bonds are 2.58% and -0.30%, respectively. Not surprisingly given the values of excess returns in real terms, no weight is given to long-term nominal bonds which are totally crowded out by TIPS for

[20]Results are available from the authors upon request.
[21]See Table 2.5.

all levels of relative risk aversion. As expected, the greatest risk-adjusted returns benefits are observed for investors with higher levels of risk aversion, ranging from $31bp$ to $248bp$, see panel i in Table 2.8. The salient point here is that buy-and-hold long-term investors will always replace long-term nominal bonds for long-term TIPS given historic real and nominal yields.

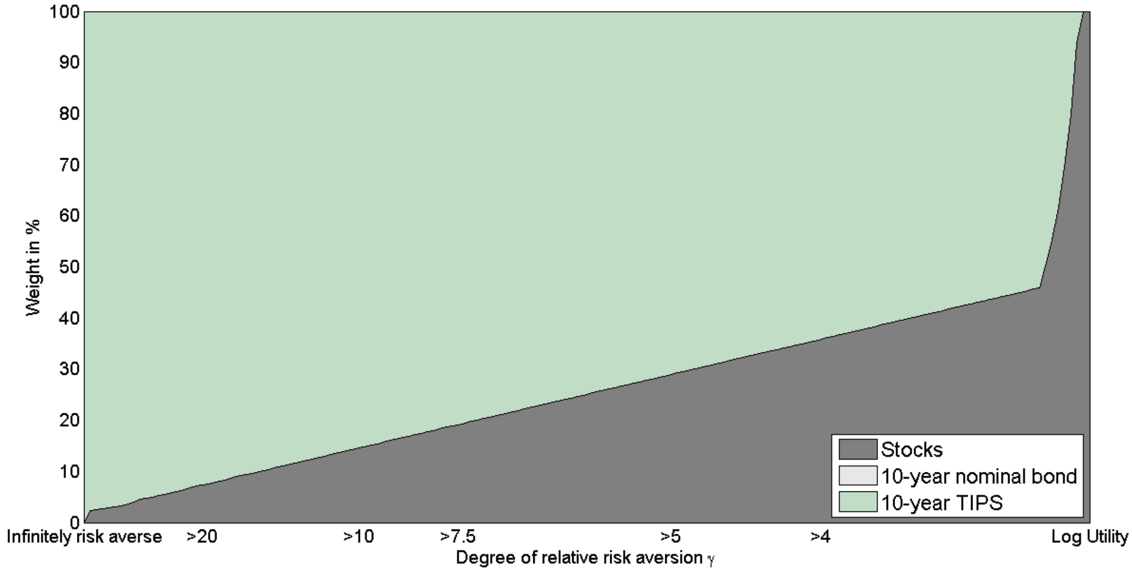


Optimal portfolio choice when a long-term investor is able to allocate her wealth into stocks, and the nominal long-term risk-free asset.

Figure 2.7: Optimal portfolio weights for a long-term investor without TIPS

Stocks, 10-year nominal, commodities, real estate, and 10-year TIPS. Panels ii-v in Table 2.8 show the benefits of introducing TIPS as a marginal security when different asset classes are available. In general, the higher the relative risk aversion, the higher the benefits that investors enjoy regardless of how wide the investment opportunity set is. As expected, infinitely risk averse investors who are not affected by money illusion allocate all their wealth to TIPS, as predicted by the theoretical model of Wachter (2003), obtaining the greatest benefits.

Our results show that when a wider set of asset classes is considered, the main benefits to long-term investors result from substituting long-term nominal bonds with TIPS. Panels iv and v in Table 2.8 show that when real estate investments are considered the benefits of investing in TIPS decrease for those investors who are not infinitely reluctant to bear risk. Intuitively, real estate investments are beneficial for long-term investors who are willing to tolerate some risk in their portfolio. For example, we see that if real estate is considered by investors with $\gamma < 10$, they will not purchase any TIPS because real estate's expected real return, corrected by risk, outperforms the real yield of TIPS. However, we note that the methodology to calculate the S&P/Case-Shiller index introduces smoothing effects that may underestimate the volatility of real estate investments.[22] Finally, investors characterized by a log utility function do not obtain any benefits from the introduction of TIPS, regardless of the investment opportunities. Indeed, they devote all their wealth to the asset with the highest real expected return.



Optimal portfolio choice when a long-term investor is able to allocate her wealth into stocks, the nominal long-term risk-free asset and TIPS which represents the real long-term risk-free asset.

Figure 2.8: Optimal portfolio weights for a long-term investor with TIPS

2.6 Concluding Remarks

In this chapter we solve an optimal portfolio choice problem in real terms in order to measure what benefits TIPS provide to investors. This approach should be used when there is uncertainty about future inflation rates and there is a riskless asset in real terms, or when there is uncertainty about future inflation rates and assets in the investment opportunity set covary with inflation. In other words, investors who are not affected by money illusion can deal with unexpected inflation rates through two possible channels: a) investing in the risk-free asset in real terms; and/or b) investing in those risky assets whose nominal returns covary with inflation.

TIPS have been issued with maturities of 5, 10, 20, and 30 years. Thus, they were primarily issued to provide a safe asset for investors with long investment horizons. The time horizon over which the investor plans to hold TIPS is relevant because only if these are

[22]We are grateful to Michael Brennan for pointing this out.

held until maturity do they behave as the riskless asset in real terms. Therefore, to better understand and measure the benefits that TIPS provide to investors we differentiate the types of investors according to their investment horizon, short-term and buy-and-hold long-term, and according to their degree of risk aversion. We measure the historical benefits that TIPS provide to the two types of investor in the presence of different asset classes such as equity, commodities, and real estate.

We find that short-term risk averse investors who are not affected by money illusion find it optimal to replace part of their investment in long-term nominal bonds with TIPS for two reasons. One, TIPS yield a slightly higher average return than nominal bonds, and two, the covariance of TIPS' returns with inflation is higher than the covariance of the returns of nominal bonds with inflation. We also find that the positive correlation of TIPS' nominal returns with inflation makes TIPS desirable for highly risk averse short-term investors since they can be used to reduce the portfolio variance in real terms. Moreover, although the relative benefits from the introduction of TIPS diminish when the short-term investor has a wider investment opportunity set which might include gold, commodities, or real estate, highly risk averse short-term investors still devote a fraction of their wealth to TIPS. Interestingly, when commodities are available, the improvement to highly risk averse short-term investors decreases because commodities are a better hedge against inflation than TIPS. Finally, short-term investors characterized by low levels of risk aversion do not obtain any benefit from the introduction of TIPS when there is a wider investment opportunity set that includes: stocks, nominal bonds, commodities, real estate, and the short-term nominal riskless asset (T-bill).

For buy-and hold long-term investors we find that: infinitely risk averse investors who are not affected by money illusion allocate all their wealth to the risk-free asset in real terms, as predicted by the theoretical model of Wachter (2003); for all levels of relative risk aversion, nominal bonds are crowded out by TIPS; when real estate is part of the investment opportunity set, the relative benefits from TIPS diminish because real estate's expected real return, corrected by risk, is high enough to outperform the real yield of TIPS; and finally, investors characterized by a log utility function do not obtain any benefits from the introduction of TIPS.

The simplicity of the portfolio choice model we consider allows us to focus on the analysis of the benefits that TIPS provide to investors. We leave for future research extensions to our model where for example one could consider dynamic rebalancing of the portfolio as well as including other instruments in the investor's opportunity set such as inflation derivatives. A model with intermediate rebalancing will allow, for example, to consider the benefits that TIPS provide to investors from a real rate hedging perspective, something which is missing in our model.

Table 2.7: Results for short-term investors

Optimal portfolio weights for short-term investors who are able (not able) to buy TIPS. Optimal weights are computed for different levels of relative risk aversion and combinations of asset classes. To illustrate the marginal benefit of TIPS we employ three different measures. The variable $RARB$ (expressed in basis points) denotes the difference between the return of the optimal portfolio which includes TIPS and the risk-adjusted expected return of the optimal portfolio, excluding TIPS. The variable RB is the ratio of the reward per unit of risk under the optimal allocation with TIPS and the reward per unit of risk under the optimal allocation without TIPS. The variable BCE is the increment of the certainty equivalent under the optimal allocation with TIPS and the certainty equivalent under the optimal allocation without TIPS.

	Optimal Weights in %											
	$\gamma = \infty$		$\gamma = 20$		$\gamma = 10$		$\gamma = 5$		$\gamma = 3$		$\gamma = 1$	
i) Stocks, 10-year nominal bonds, 10-year TIPS and T-bills												
Stocks	1.3	(1.3)	14.6	(16.8)	27.8	(32.7)	41.7	(45.0)	58.4	(58.5)	100	(100)
10-year nominal bonds	0.0	(0.0)	9.1	(29.8)	26.1	(62.2)	33.0	(55.0)	41.3	(41.5)	0.0	(0.0)
10-year TIPS	3.5	(0.0)	32.0	(0.0)	46.1	(0.0)	25.3	(0.0)	0.3	(0.0)	0.0	(0.0)
T-bills	95.2	(98.7)	44.4	(53.3)	0.0	(5.1)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
RARB	12bp		16bp		28bp		19bp		0bp		0bp	
RB	1.086		1.047		1.056		1.034		1.000		1.000	
BCE	n.a.		0.17%		0.20%		0.03%		≈ 0%		0%	
ii) Stocks, 10-year nominal bonds, 10-year TIPS, Gold and T-bills												
Stocks	1.3	(1.3)	14.7	(16.6)	26.9	(30.8)	41.2	(43.3)	58.3	(58.3)	100	(100)
10-year nominal bonds	0.0	(0.0)	11.6	(29.8)	27.5	(56.5)	33.8	(49.4)	40.9	(40.9)	0.0	(0.0)
10-year TIPS	3.1	(0.0)	28.1	(0.0)	36.2	(0.0)	19.5	(0.0)	0.0	(0.0)	0.0	(0.0)
Gold	1.5	(1.7)	6.4	(7.8)	9.4	(12.7)	5.5	(7.3)	0.8	(0.8)	0.0	(0.0)
T-bills	94.1	(97)	39.2	(45.7)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
RARB	10bp		12bp		22bp		11bp		0bp		0bp	
RB	1.071		1.035		1.045		1.020		1.000		1.000	
BCE	n.a.		0.13%		0.12%		0.02%		0%		0%	
iii) Stocks, 10-year nominal bonds, 10-year TIPS, Commodities and T-bills												
Stocks	0.2	(0.2)	13.9	(14.9)	26.9	(29.9)	40.4	(41)	54.3	(54.3)	100	(100)
10-year nominal bonds	0.0	(0.0)	16.4	(30.0)	35.3	(61.6)	45.4	(50.3)	36.7	(36.7)	0.0	(0.0)
10-year TIPS	0.0	(0.0)	20.9	(0.0)	31.7	(0.0)	5.9	(0.0)	0.0	(0.0)	0.0	(0.0)
Commodities	3.8	(3.8)	4.8	(6.2)	6.1	(8.5)	8.3	(8.8)	9.0	(9.0)	0.0	(0.0)
T-bills	96.0	(96.0)	44.0	(48.9)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
RARB	0bp		4bp		17bp		3bp		0bp		0bp	
RB	1.000		1.013		1.033		1.006		1.000		1.000	
BCE	n.a.		0.06%		0.08%		≈ 0%		0%		0%	
iv) Stocks, 10-year nominal bonds, 10-year TIPS, Real estate and T-bills												
Stocks	1.3	(1.3)	9.6	(10.8)	17.1	(17.7)	31.6	(31.6)	48.3	(48.3)	100	(100)
10-year nominal bonds	0.0	(0.0)	7.3	(18.9)	11.5	(17.4)	14.3	(14.3)	10.7	(10.7)	0.0	(0.0)
10-year TIPS	3.5	(0.0)	16.7	(0.0)	8.4	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
Real estate	1.3	(1.4)	66.5	(70.3)	63.0	(64.9)	54.0	(54.0)	41.0	(41.0)	0.0	(0.0)
T-bills	94.0	(97.4)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
RARB	12bp		13bp		7bp		0bp		0bp		0bp	
RB	1.081		1.028		1.013		1.000		1.000		1.000	
BCE	n.a.		0.05%		≈ 0%		0%		0%		0%	
v) All Assets												
Stocks	0.2	(0.2)	9.3	(9.8)	16.8	(16.8)	30.5	(30.5)	46.9	(46.9)	100	(100)
10-year nominal bonds	0.0	(0.0)	11.7	(18.1)	17.4	(17.4)	15.1	(15.1)	11.6	(11.6)	0.0	(0.0)
10-year TIPS	0.0	(0.0)	9.2	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
Gold	0.0	(0.0)	4.1	(4.5)	2.0	(2.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
Commodities	3.8	(3.8)	2.1	(2.6)	3.6	(3.6)	5.0	(5.0)	6.3	(6.3)	0.0	(0.0)
Real estate	0.0	(0.0)	63.5	(65.0)	60.3	(60.3)	49.5	(49.5)	35.3	(35.3)	0.0	(0.0)
T-bills	96.0	(96.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
RARB	0bp		5bp		0bp		0bp		0bp		0bp	
RB	1.000		1.011		1.000		1.000		1.000		1.000	
BCE	n.a.		≈ 0%		0%		0%		0%		0%	

n.a.: not applicable.

Table 2.8: Results for long-term investors

Optimal portfolio weights for long-term investors who are able (not able) to buy TIPS. Optimal weights are computed for different levels of relative risk aversion and combinations of asset classes. To illustrate the marginal benefit of TIPS we employ three different measures. The variable *RARB* (expressed in basis points) denotes the difference between the return of the optimal portfolio which includes TIPS and the risk-adjusted expected return of the optimal portfolio, excluding TIPS. The variable *RB* is the ratio of the reward per unit of risk under the optimal allocation with TIPS and the reward per unit of risk under the optimal allocation without TIPS. The variable *BCE* is the increment of the certainty equivalent under the optimal allocation with TIPS and the certainty equivalent under the optimal allocation without TIPS.

	Optimal Weights in %											
	$\gamma = \infty$		$\gamma = 20$		$\gamma = 10$		$\gamma = 5$		$\gamma = 3$		$\gamma = 1$	
i) Stocks, 10-year nominal bonds and 10-year TIPS												
Stocks	0.0	(1.3)	7.1	(8.8)	14.3	(16.4)	28.7	(31.5)	45.9	(49.7)	100	(100)
10-year nominal bonds	0.0	(98.7)	0.0	(91.2)	0.0	(83.6)	0.0	(68.5)	0.0	(50.3)	0.0	(0.0)
10-year TIPS	100	(0.0)	92.9	(0.0)	85.7	(0.0)	71.3	(0.0)	54.1	(0.0)	0.0	(0.0)
RARB	248bp		112bp		66bp		41bp		31bp		0bp	
RB	∞		1.684		1.276		1.129		1.079		1.000	
BCE	∞		0.39%		0.28%		0.19%		0.10%		0%	
ii) Stocks, Gold, 10-year nominal bonds and 10-year TIPS												
Stocks	0.0	(1.3)	7.1	(8.6)	14.2	(16.1)	28.6	(31.0)	45.8	(48.9)	87.1	(87.1)
Gold	0.0	(1.7)	4.8	(6.9)	9.6	(12.2)	19.2	(22.8)	30.8	(35.4)	12.9	(12.9)
10-year nominal bonds	0.0	(97.0)	0.0	(84.4)	0.0	(71.7)	0.0	(46.3)	0.0	(15.7)	0.0	(0.0)
10-year TIPS	100	(0.0)	88.1	(0.0)	76.2	(0.0)	52.2	(0.0)	23.4	(0.0)	0.0	(0.0)
RARB	248bp		99bp		57bp		35bp		23bp		0bp	
RB	∞		1.536		1.216		1.096		1.051		1.000	
BCE	∞		0.34%		0.22%		0.10%		$\approx$ 0%		0%	
iii) Stocks, Commodities, 10-year nominal bonds and 10-year TIPS												
Stocks	0.0	(0.1)	6.8	(7.3)	13.7	(14.5)	27.6	(29.0)	44.2	(46.4)	93.5	(93.5)
Commodities	0.0	(3.8)	1.3	(4.9)	2.5	(6.0)	5.1	(8.2)	8.1	(10.8)	6.5	(6.5)
10-year nominal bonds	0.0	(96.1)	0.0	(87.8)	0.0	(79.5)	0.0	(62.8)	0.0	(42.8)	0.0	(0.0)
10-year TIPS	100	(0.0)	91.9	(0.0)	83.7	(0.0)	67.4	(0.0)	47.7	(0.0)	0.0	(0.0)
RARB	248bp		80bp		46bp		30bp		23bp		0bp	
RB	∞		1.406		1.175		1.090		1.056		1.000	
BCE	∞		0.27%		0.20%		0.13%		0.06%		0%	
iv) Stocks, Real estate, 10-year nominal bonds and 10-year TIPS												
Stocks	0.0	(1.2)	5.1	(5.7)	9.5	(9.5)	16.9	(16.9)	25.8	(25.8)	75.9	(75.9)
Real estate	0.0	(1.9)	60.8	(77.4)	90.5	(90.5)	83.1	(83.1)	74.2	(74.2)	24.1	(24.1)
10-year nominal bonds	0.0	(96.9)	0.0	(16.9)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
10-year TIPS	100	(0.0)	34.1	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
RARB	248bp		60bp		0bp		0bp		0bp		0bp	
RB	∞		1.183		1.000		1.000		1.000		1.000	
BCE	∞		0.11%		0%		0%		0%		0%	
v) All Assets												
Stocks	0.0	(0.1)	4.8	(5.1)	8.9	(8.9)	16.4	(16.4)	25.4	(25.4)	75.9	(75.9)
Gold	0.0	(0.0)	6.5	(8.1)	8.3	(8.3)	7.2	(7.2)	5.9	(5.9)	0.0	(0.0)
Commodities	0.0	(3.8)	0.6	(1.1)	0.8	(0.8)	0.7	(0.7)	0.5	(0.5)	0.0	(0.0)
Real estate	0.0	(0.0)	63.3	(78.2)	82.0	(82.0)	75.8	(75.8)	68.2	(68.2)	24.1	(24.1)
10-year nominal bonds	0.0	(96.0)	0.0	(7.5)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
10-year TIPS	100	(0.0)	24.7	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)	0.0	(0.0)
RARB	248bp		48bp		0bp		0bp		0bp		0bp	
RB	∞		1.136		1.000		1.000		1.000		1.000	
BCE	∞		0.05%		0%		0%		0%		0%	

Chapter 3

The U.S. Breakeven Inflation Rates in a Preferred-Habitat Model

3.1 Introduction

The difference in yields between a zero-coupon government nominal bond and an inflation-linked (IL) bond with the same maturity is known as the breakeven inflation rate (BEI).[1] While a nominal bond with maturity τ offers a riskless nominal yield to buy-and-hold investors with an investment horizon τ, an IL bond with the same maturity τ provides a riskless real yield to these buy-and-hold investors.[2] Ex-post, the BEI represents the average realized inflation rate over τ that equates the buy-and-hold return of both types of bonds. Ex-ante, it is argued that the BEI should be equal to the average expected inflation rate over τ plus a positive inflation risk premium since risk-averse buy-and-hold investors will be willing to pay a positive premium to buy IL instead of nominal bonds for the protection they will receive against inflation risk.

One of the reasons why a government should issue IL bonds instead of nominal ones is because by issuing IL bonds a government may reduce borrowing costs by not having to pay the inflation risk premium. Figure 3.1 exhibits the U.S. Treasury 10-year BEI and the U.S. 10-year-ahead inflation forecasts from the Survey of Professional Forecasters (SPF). If the inflation risk premium is positive, the BEI should be above the expected inflation rate. In contrast, the Figure shows that in the years before 2004 and also just after the collapse of Lehman Brothers the BEI was far below the expected inflation rate. Related to this "anomaly" in the U.S. BEI, Fleckenstein et al. (2010) present the TIPS-Treasury bond puzzle by showing that TIPS are consistently undervalued relative to nominal U.S. Treasury bonds.[3] In other words, they show that the U.S. Treasury has increased its financing cost by launching the Treasury Inflation Protected Securities (TIPS) program.[4]

The common explanation for the TIPS-Treasury bond puzzle is because of a liquidity premium in TIPS, as explained in Sack and Elsasser (2004), Roush (2008), Dudley et al. (2009), D'Amico et al. (2010), Fleming and Krishnan (2009), Viceira and Pflueger (2011), among others. However, if IL bonds are bought by buy-and-hold investors with an investment horizon of the same length as the maturity of the bond, they should not be concerned about the poor liquidity in the secondary market of these instruments relative to nominal bonds. Indeed, buy-and-hold investors will be better off by buying IL bonds with an extra liquidity premium. For instance, Vayanos and Vila (1999) show that for a long holding period transaction costs are less important and illiquid assets are the better investment. In contrast, for a short investment holding period transaction costs are relevant and liquid assets are the better investment even though they are more expensive

[1] Zero-coupon government bonds will be simply referred to as "bonds" for the remainder of the chapter.

[2] See Campbell and Viceira (2001), Brennan and Xia (2002), Campbell et al. (2003) and Wachter (2003).

[3] They provide empirical evidence that the TIPS mispricing is strongly influenced by supply factors. Particularly, they find the size of the mispricing decreases significantly when the Treasury issues either Treasury bonds or TIPS, and increases with the amount of repo failures in the primary dealer market.

[4] TIPS are IL bonds issued by the U.S. Treasury. The primary feature of TIPS is that their principal is indexed to the U.S. non-seasonally adjusted consumer price index for all urban consumers (CPI-U NSA).

than illiquid assets.

An alternative explanation for the puzzle is related to the preferred-habitat hypothesis (PHH) proposed by Modigliani and Sutch (1966). The main implication of this hypothesis over the term structure of interest rate is that it implies that demand and supply factors affect bonds' yields which is an aspect that is absent in standard no-arbitrage term structure models.[5] In a recent article, Vayanos and Vila (2009) formalize the preferred-habitat hypothesis of one specific type of debt in a way consistent with the no-arbitrage condition in which they address the effects of demand shocks over the cross-section of maturities.

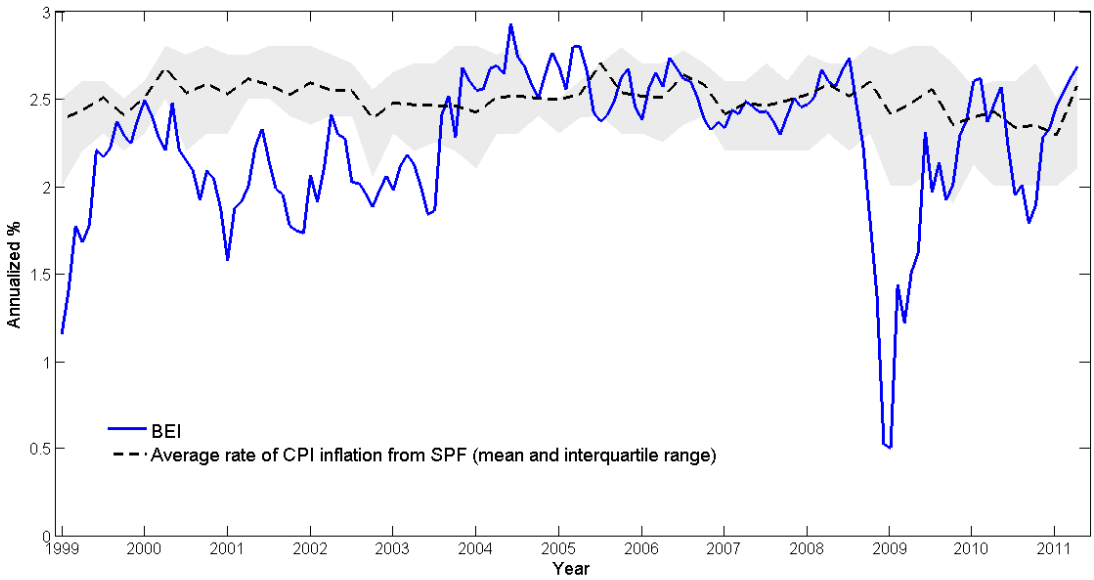
The PHH combines the ideas of three different theories about the determinants of the yield term structure: the pure expectation hypothesis (PEH) by Fisher (1896) and Lutz (1940); the liquidity preference hypothesis (LPH) by Hicks (1939); and the market segmentation hypothesis (MSH) by Culbertson (1957).[6] The PEH states that the term structure of interest rate is driven by the investors' expectations on futures spot rates and all term premiums are zero which implies that the BEI should be equal to the average expected inflation rate. On the other hand, the LPH predicts that the yield curve will be above the curve implied by the PEH because of the uncertainty of futures rates the yield curve will include an increasing risk premium as the term to maturity increases. In this case, the BEI should be equal to the average expected inflation rate plus a positive and increasing inflation risk premium. Finally, the MSH which states that investors have definite preferences for instruments of a specific type and maturity. Thus, the nominal and real yields for nominal bonds and IL bonds will be determined in separate market, by their own independent supply and demand factors. In this case, the BEI will not necessarily need to be related to the average expected inflation rate.

The PHH proposes that investors have specific maturity preferences as the MSH, but with some degree of substitution across types of bonds and maturity segments if the reward is high enough. This hypothesis agrees with the LPH that the yield curve is controlled by the average futures rates but modified by the risk premiums. However, the LPH assumes that investors are concerned to turn their bond portfolio back into cash in a short period and then the bond's premium will be an increasing function on the term to maturity. In contrast, the PHH states that investors have different habitats and then the yield curve may be above or below the curve implied by the PEH as a consequence of positive or negative risk premiums. Under this hypothesis, the BEI should be equal to the average expected inflation rate plus a positive or negative inflation risk premium, a fact that is confirmed in Figure 3.1.

In this chapter we study a theoretical BEI term structure in a preferred-habitat framework. What we propose is a theoretical model in which investors have specific preferences for one type of bond (IL or nominal) but with some degree of substitution across them. The model assumes that bonds' yields come from the interaction between buy-and-hold

[5]See Cox et al. (1985), Duffie and Kan (1996), Duffee (2002) and Piazzesi (2005).
[6]See Modigliani and Sutch (1966).

or preferred-habitat investors (PHI) who demand IL bonds with specific maturity and risk-averse arbitrageurs or mark-to-market investors who allocate their real wealth in IL bonds and nominal bonds based on an instantaneous mean-variance problem. We extend Vayanos and Vila (2009) by adding inflation into the model and by distinguishing between the real and the nominal term structure of interest rates. Our objective is to understand the BEI components when there exists a certain degree of market segmentation and whether this framework is able to explain the U.S. BEI puzzle.



The 10-year BEI is from Gürkaynak et al. (2007, 2010). The projections of the 10-year annual average rate of CPI inflation is from the Survey of Professional Forecasters.

Figure 3.1: U.S. 10-year breakeven inflation rate and forecast of CPI inflation

We develop the basic model in which there are two risk factors: the real interest rate and the inflation rate which follow two correlated Onstein-Uhlenbeck processes. The model states that market prices of risk or equivalently bonds' yields are determined by the total exposure of arbitrageurs' portfolio to each source of risk. We distinguish between a direct exposure and a cross-effect in which the later depends on the correlation between the real short rate and the inflation rate. Specifically, the market price of real interest rate risk is given by the direct exposure for arbitrageurs' holdings of IL bonds and nominal bonds and by the cross-effect for arbitrageurs' holdings of nominal bonds. Similarly, the market price of inflation risk is given by the direct exposure for arbitrageurs' holdings of nominal bonds and by the cross-effect for arbitrageurs' holdings of IL bonds and nominal bonds.

In the absence of PHI, the model is equivalent to the two-factor Vasicek model in which market prices of risk are determined by the arbitrageurs' risk aversion coefficient and the total outstanding level of IL debt and nominal debt relative to the arbitrageurs' level of real wealth.[7] In this context, the term structure of BEI is driven by the expectations on future instantaneous inflation rates plus an inflation risk premium that depends on the maturity of the bonds but not on time. The main result is that the inflation risk premium can be either positive or negative. This result is in contrast to the common agreement that the BEI should be equal to the average expected inflation rate over τ plus a positive inflation risk premium. For instance, when the main source of risk of arbitrageurs' portfolio is given by the real interest rate risk and the correlation between the inflation and the real interest rate is negative, the inflation uncertainty will behave as a hedging component in arbitrageurs' portfolios.

In the model we assume that the demand of PHI for IL bonds is an increasing function of IL bonds' real yields and a decreasing function of the real return of alternative investment opportunities for them. When PHI participate in the IL bond market and arbitrageurs are risk-averse, the model establishes that the BEI is an affine function not only of the expected rate of inflation but also of the real short rate if the inflation and the real short rate are correlated. Under these assumptions, the inflation risk premium will be time-varying, that is, it will depend on both the term to maturity and the current time. The intuition is as follows. In equilibrium, arbitrageurs' holdings of IL bonds complement the demand of PHI to clear the market. Time-varying real yields implies that PHI will change their demand of IL bonds affecting arbitrageurs holdings of IL bonds. Then, if the correlation between the inflation and the real short rate is different from zero the BEI term premium will be affected by the cross-effect on the inflation market price of risk for arbitrageurs' holdings of IL bonds.

Vayanos and Vila (2009) show that in the presence of PHI forward rates under-react to changes in expected spot rates. That is, risk-averse arbitrageurs partially incorporate information about expected short rate into forward rates and the degree of under-reaction depends on the arbitrageurs' level of risk aversion. We find that forward nominal rates react differently than real forward rates to changes in the real short rate when the inflation and the real short rate are correlated. This implies that when the risk aversion of arbitrageurs is high, the forward BEI may not adequately capture the compensation that buy-and-hold investors demand to cover the expected rate of inflation. That is, the forward BEI will include the compensation that investors demand for expected inflation, the risk associated with that inflation, and a term proportional to the real short rate that affects the market price of inflation through the cross-effect given by the correlation coefficient.

In this chapter we obtain market prices of risk as a function of the total outstanding debt of IL and nominal bonds, and the demand of IL securities made by PHI relative to arbitrageurs' real wealth. This representation allows us, for example, to understand the effect of debt issued by Treasuries or open market operations carried by central banks on market prices of risk. We find that the market price of risk for the real interest rate is

[7] See Vasicek (1977) and Chen (1995).

positively affected by an increase in the supply of IL bonds. We also obtain that a change in the supply of IL bonds or in the real return of alternative investment opportunities for PHI affect the inflation risk premium when the real interest rate and the inflation are correlated. Finally, we get that the effect of an increase in the supply of nominal bonds on market prices of risk depends on the correlation between the real interest rate and the inflation rate.

Our preferred-habitat model provides implications about the benefits of issuing IL bonds instead of conventional nominal ones.[8] First, it is argued that by issuing IL bonds instead of nominal ones with similar maturity a government may reduce borrowing costs by not having to pay the inflation risk premium. We find that when the main source of risk of arbitrageurs' portfolio is given by the real interest rate and the correlation between the inflation and the real rate is negative, the inflation exposure will hedge the real rate risk of arbitrageurs' portfolio and the inflation uncertainty will show a negative risk premium. Thus, the common advice that by issuing IL bonds a government can save the inflation risk premium is not always true.

Second, by issuing different types of debt a government can obtain diversification benefits of its financial debt position. The standard no-arbitrage term structure models used to reproduce the dynamics of real and nominal yields are based on the assumption of a representative agent model where demand and supply effects do not play any role. In contrast, the preferred-habitat model provides information about how bond demand and supply changes affects IL and nominal yields. We find that the issuance of IL bonds will help the Treasury for diversification purposes under two different conditions. First, when the correlation between the real rate and the inflation is non-negative. And second, when the main source of risk of arbitrageurs' portfolio is given by the inflation risk, and there is a negative correlation between the real interest rate and the inflation.

Third, the BEI is used to extract instantaneous market information about long-term inflation expectations which contributes to enhance the effectiveness of central banks' decisions. We find that the forward BEI under the preferred-habitat model includes the expected inflation rate, the inflation risk premium, and a term proportional to the real short rate. The last term appears as a consequence of the correlation between the inflation rate and the real interest rate and it reflects the time-varying behavior of the inflation risk premium. Then, in periods of financial distress when arbitrageurs are more risk averse, the forward BEI may not adequately capture the compensation that buy-and-hold investors demand to cover the expected rate of inflation since arbitrageurs care about short-term bond returns instead of long-term bond yields.

The remainder of this chapter is structured as follows. Section 3.2 presents some empirical facts about the U.S. BEI and descriptive statistics of investor class allotment of nominal and IL U.S. bonds. Section 3.3 derives the basic model in which the real

[8] See Shen (1995), Barr and Campbell (1997), Deacon et al. (2004) , and Dudley et al. (2009) for IL bond benefits.

short rate and inflation follow two correlated mean-reverting processes. We assume that demand factors are constant and that the government adjusts the total outstanding debt of nominal and IL bonds to changes in the arbitrageurs' level of real wealth. Section 3.4 exposes and analyses the theoretical term structure of breakeven inflation rates predicted by the model. Section 3.5 discusses implications of the model. Section 3.6 concludes.

3.2 Stylized Facts

3.2.1 The U.S. breakeven inflation rates

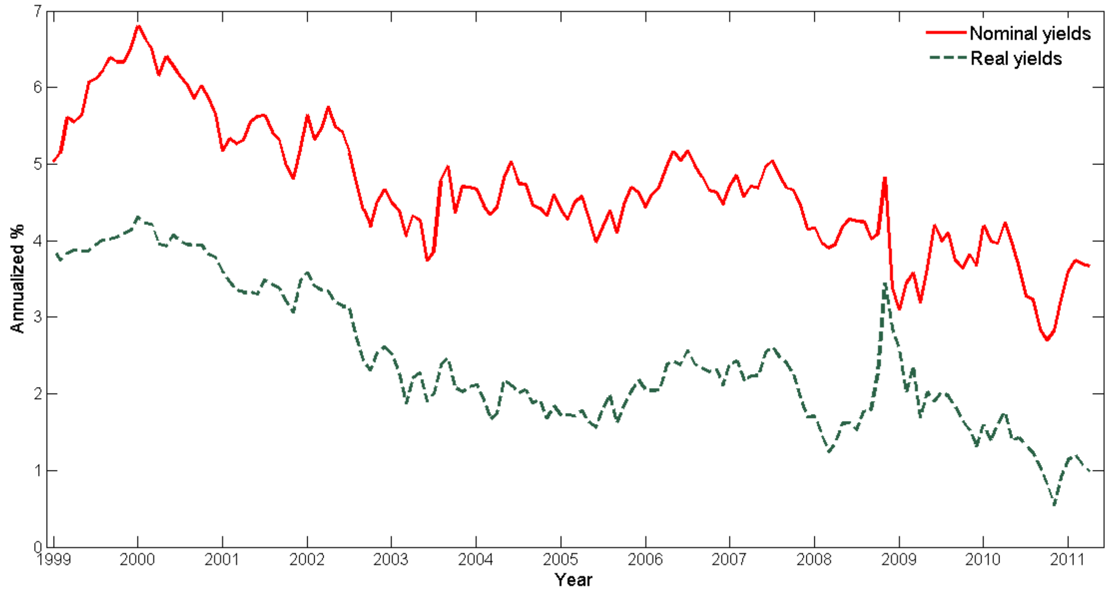
The BEI is commonly used to obtain an approximation of the market participants' average expected rate of inflation from today until the maturity of the bonds. However, there are some shortcomings with this simple methodology to get the inflation expectations of market participants. First, it can be difficult to find a conventional nominal and an IL bond issued with exactly the same maturity dates. Second, there are problems caused by imperfect indexation.[9] Third, estimates of inflation expectations will be distorted by the risk premiums included in real and nominal yields.

To overcome the first shortcoming, Gürkaynak et al. (2007, 2010) fit a nominal and real yield curve to U.S. Treasury securities, both nominal and IL bonds, that allows measures of inflation compensation to be computed for any horizon.[10] Figure 3.2 shows the U.S. 10-year nominal and real yields constant maturity from January 1999 to May 2011. Both time series refer to the yield of zero-coupon bonds which means that the nominal bond offers a riskless nominal yield to buy-and-hold investors with a 10-year investment horizon while the IL bond provides them a riskless real yield.

The 10-year BEI constant maturity is plotted in Figure 3.1 together with the projections of the 10-year annual average rate of inflation from SPF. Risk-averse buy-and-hold investors will be willing to pay a positive premium to buy IL bonds instead of nominal bonds for the protection they will receive against inflation risk. For this reason, one would expect to see the BEI always above the expected rate of inflation reflecting a positive inflation risk premium. However, Figure 3.1 shows that in the years before 2004 and months after the collapse of Lehman (2008M10-2010M1), the BEI is far below the expected rate of inflation.

[9] To mention some: a) market participants' basket might differ from the basket used to calculate the index to which the IL is indexed; b) there is usually a lag in the indexation rule; c) there are tax considerations; and, d) there is reinvestment risk arising from the coupon flows received before maturity.

[10] They used a well-known Nelson-Siegel-Svensson approach which is a simple smoothing method that is shown to fit the data very well, see Nelson and Siegel (1987) and Svensson (1994).



The US 10-year nominal and real yields constant maturity for the period January 1999 to May 2011 from Gürkaynak et al. (2007, 2010).

Figure 3.2: U.S. 10-year nominal and real yields

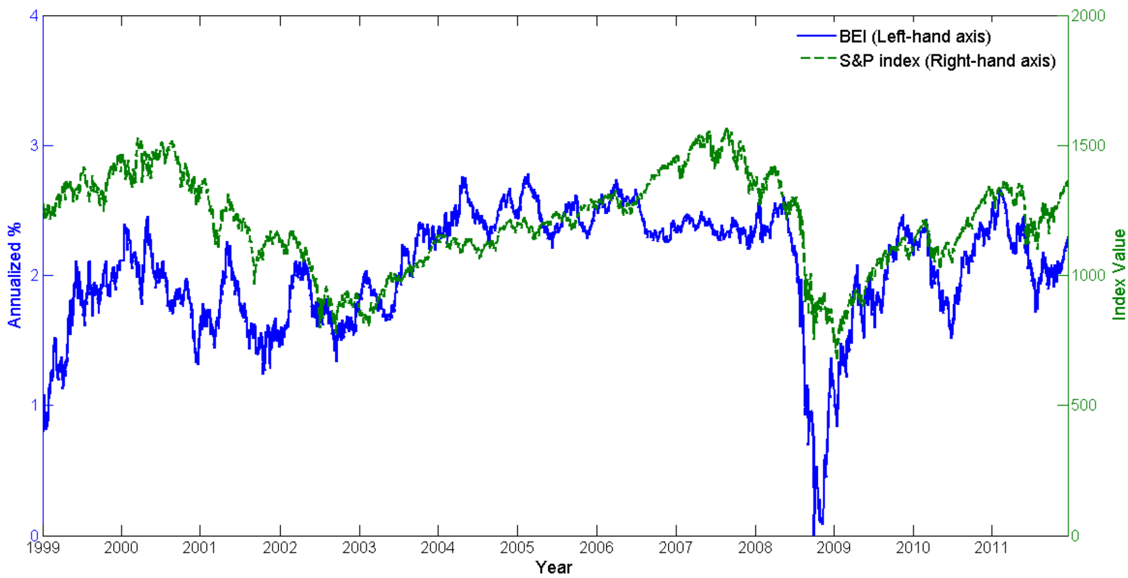


Figure 3.3: U.S. 10-year BEI and S&P index

Figure 3.3 plots the 10-year BEI and the S&P 500 Index. The Figure shows a positive relation between both variables which is confirmed by the correlation coefficient of 0.46 for the complete sample. The relation between both variables increases in the last part of the sample observing a correlation coefficient of almost 0.80 when we use data after 2007.

This positive relation between the BEI and and stock index is also confirmed when we consider other countries like U.K., Germany, Italy and Australia for instance. In terms of market volatility, the BEI seems to be negatively related with market uncertainty. Figure 3.4 exhibits the negative relationship between the 10-year BEI and the VIX Index with a correlation coefficient of -0.76 for the complete sample.

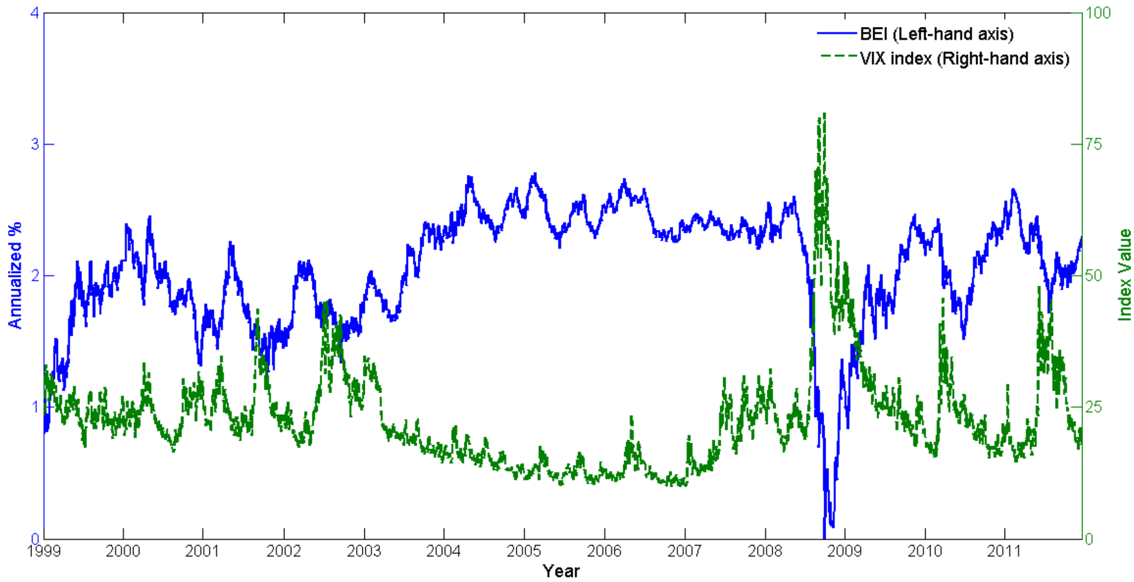
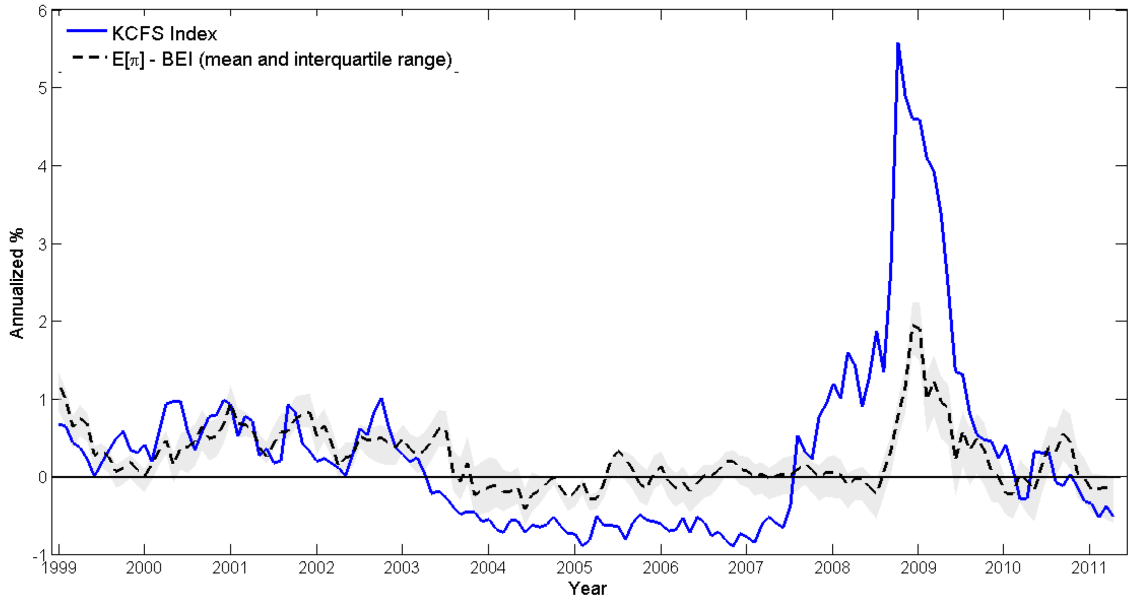


Figure 3.4: U.S 10-year BEI and VIX index

Figure 3.5 plots the 10-year BEI Premia and Kansas City Financial Stress (KCFS) Index. We define the 10-year BEI Premia as the difference between the 10-year forward inflation (mean and interquartile range) from SPF and the 10-year BEI. In the "normal" case we would expect to observe a negative value reflecting a positive inflation risk premium, that is the BEI will be the sum of the expected inflation plus a positive inflation risk premium. In contrast, a positive value will refer to the existence of a negative premium contained in the BEI. The KCFS Index is a monthly measure of stress in the U.S. financial system based on 11 financial market variables.[11] A positive value indicates that financial stress is above the long-run average, while a negative value signifies that financial stress is below the long-run average. The Figure shows that positive values of the 10-year BEI Premia are generally associated with financial stress above the long-run average, years before 2004 and months after the collapse of Lehman Brothers. During these two periods TIPS were cheaper relative to nominal bonds for buy-and hold investors, see Fleckenstein et al. (2010). On the other hand, in periods when financial stress is below its long-run average it seems that the BEI properly capture the compensation that buy-and-hold investors demand to cover the expected rate of inflation.

[11]Fore more details about the index construction see Hakkio and Keeton (2009).



KCFS Index is a monthly measure of stress in the U.S. financial system based on 11 financial market variables. A positive value indicates that financial stress is above the long-run average, while a negative value signifies that financial stress is below the long-run average. The BEI premia is obtained by subtracting the 10-year BEI from the 10-year forward inflation (mean and interquartile range).

Figure 3.5: KCFS index and the BEI premia

3.2.2 Who buys IL at auctions?

One may argue that during periods of financial stress market participants desire highly safe and liquid securities. However, the riskiness and liquidity feature of assets depends on the type of investor and investment horizon, see for instance Vayanos and Vila (1999) and Campbell and Viceira (2005). In this section we provide a comparative analysis about nominal and IL bond buyers that participate in auctions. As pointed by Fleming (2007),[12] although bonds' ownership can change over time, we consider that the allocation of an issue at auction provides useful information about the class of investor who demand each type of bond and the required compensation. We focus on the 10-year segment because TIPS of this maturity are the only ones who have been continuously issued since 1997.

Table 3.1 reports descriptive statistics of investor class allotment shares at auction for 10-year nominal and IL U.S. Treasury security issued between January 18, 2000 and May 31, 2011. The Table shows that dealers and brokers represent, on average, the largest investor class bond buyer in both nominal and IL with a higher share in the

[12]Fleming (2007) provides a descriptive analysis of U.S. bond buyers at auctions and evaluates the extent to which the indirect bid is a good proxy for purchases by foreign investors as a group.

former, 67.65% against 57.49% in the IL auctions. Dealers and brokers' smaller share of IL bond purchases relative to nominal purchases is compensated by higher IL bond shares for investment funds which represent the second largest group with 27.48%. Together, dealers and brokers and investment funds account for 81.78% of nominal securities and 84.97% of IL bonds sold to the public. Foreign and international investors account for the second largest share in nominal bonds (16.69%) followed by investment funds (14.13%) while Individuals and pension funds together represent less than 0.70%, on average. In the case of IL bonds, foreign and international investors purchase smaller shares of securities (10.56%) than of nominal bonds, with investment funds, individuals (2.23%) and pension funds (1.03%) covering the difference.[13]

Table 3.1 also reports descriptive statistics of two subperiods where TIPS were cheaper relative to nominal bonds for buy-and hold investors and financial stress was above the long-run average. Previous to 2004, dealers and brokers and investment funds accounted for 84.15% of IL securities which is almost exactly the same as for the whole sample but with a slightly higher standard deviation. The average share of foreign and international investors presents a lower value in this subperiod while individuals and pension funds more than double their average share in comparison with the whole sample. In the case of nominal bonds, the Table shows an increase in the participation of dealers and brokers and investment funds of almost 4%, mainly explained by dealers and brokers, together with a reduction in the standard deviation. In contrast, foreign and international investors reduce their participation in more than 5% in average with a huge decrease in the standard deviation. At the same time, individuals increase their average value but with an increase in the standard deviation, and pension funds exhibit the same negligible value as of the complete sample.

When we consider auctions after the collapse of Lehman Brothers, from October 2008 up to January 2010, we observe a reduction in the average share of dealers and brokers and investment funds together of more than 7% of IL securities, mainly explained by investment funds. Individuals and pension funds present higher share values than those of the complete sample with an important decrease in the standard deviation. Foreign and international investors also exhibit an increase in the average share together with a slightly increase in the dispersion. At the same time, dealers and brokers and investment funds increase their average share of nominal securities from 81.78% in the complete sample to 84.53% with a tiny decrease in the standard deviation. In contrast, compared to the complete sample, foreign and international investors, individuals and pension funds reduce the average and the standard deviation of their share of nominal bonds. Particularly, individuals and pension funds account for 0.29% of nominal securities while they represent 4.98% of IL securities in this subsample.

[13]Pension funds refer to the category "Pension and Retirement Funds and Insurance Companies".

Table 3.1: Investor class allotment shares for 10-year nominal and IL Treasury securities

Investor class: (1) Total issue net SOMA (in billons of dollars); **(2)** Depository Institutions; **(3)** Individuals; **(4)** Dealers and Brokers; **(5)** Pension and Retirement Funds and Insurance Companies; **(6)** Investment Funds; **(7)** Foreign and International; **(8)** Other. The table reports descriptive statistics of investor class allotment shares in percent for 10-year nominal and IL U.S. Treasury security auctions between January 18, 2000 and May 31, 2011. Source: Authors' calculations, based on data from the U.S. Treasury Department.

Investor Class	**(1)**	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**	**(8)**	**(3) + (5)**	**(4) + (6)**
I. 10-year Nominal Treasury Notes										
a. All sample - From 2000 up to May 2011 ($n = 92$)										
Mean	14.83	0.70%	0.60%	67.65%	0.09%	14.13%	16.69%	0.15%	0.69%	81.78%
Std	5.35	1.72%	0.72%	14.67%	0.39%	8.66%	11.04%	0.25%	0.80%	11.01%
Min	6.00	0.00%	0.00%	29.81%	0.00%	0.28%	0.34%	0.00%	0.00%	46.26%
Max	25.00	12.23%	4.40%	97.09%	2.88%	46.10%	53.12%	1.35%	4.40%	99.11%
b. Before 2004 ($n = 19$)										
Mean	12.59	1.55%	1.29%	73.70%	0.05%	11.91%	11.21%	0.29%	1.34%	85.61%
Std	4.06	1.80%	1.16%	8.12%	0.21%	7.21%	4.58%	0.35%	1.19%	4.56%
Min	6.00	0.01%	0.00%	54.80%	0.00%	0.34%	2.92%	0.00%	0.00%	78.55%
Max	18.01	6.42%	4.40%	88.82%	0.91%	28.22%	19.85%	1.22%	4.40%	96.21%
c. Months after the collapse of Lehman Brothers - from October 2008 to January 2010 ($n = 19$)										
Mean	17.84	0.44%	0.24%	65.91%	0.05%	18.62%	14.67%	0.07%	0.29%	84.53%
Std	4.68	0.95%	0.24%	12.05%	0.08%	7.21%	9.89%	0.10%	0.26%	10.44%
Min	10.00	0.00%	0.00%	45.58%	0.00%	6.14%	1.30%	0.00%	0.00%	67.69%
Max	25.00	4.18%	0.97%	87.80%	0.26%	31.27%	31.17%	0.31%	0.97%	98.68%
II. 10-year U.S. TIPS										
a. All sample - From 2000 up to May 2011 ($n = 42$)										
Mean	8.36	0.80%	2.23%	57.49%	1.03%	27.48%	10.56%	0.41%	3.26%	84.97%
Std	2.00	2.55%	2.06%	9.62%	1.76%	9.78%	6.22%	0.89%	3.14%	6.58%
Min	5.00	0.00%	0.26%	38.73%	0.00%	9.25%	0.38%	0.00%	0.33%	70.52%
Max	13.00	12.59%	8.91%	81.37%	9.44%	48.61%	24.73%	4.35%	13.89%	95.97%
b. Before 2004 ($n = 10$)										
Mean	7.00	1.23%	4.72%	56.15%	2.25%	28.00%	6.44%	1.21%	6.97%	84.15%
Std	2.00	3.13%	2.56%	10.98%	3.16%	7.75%	5.96%	1.56%	3.97%	7.62%
Min	5.00	0.01%	1.13%	42.52%	0.07%	16.81%	0.38%	0.04%	1.83%	74.28%
Max	11.01	10.03%	8.91%	73.63%	9.44%	38.91%	18.55%	4.35%	13.89%	95.97%
c. Months after the collapse of Lehman Brothers - from October 2008 to January 2010 ($n = 6$)										
Mean	7.49	0.13%	3.07%	55.27%	1.91%	22.46%	16.79%	0.36%	4.98%	77.74%
Std	1.52	0.08%	0.84%	11.36%	0.74%	12.02%	7.02%	0.39%	0.97%	6.70%
Min	5.98	0.06%	1.98%	38.73%	1.00%	11.80%	8.66%	0.06%	3.85%	70.52%
Max	10.00	0.29%	4.13%	73.69%	2.86%	41.54%	24.73%	0.95%	6.17%	85.49%

3.3 The Model

We consider a continuous time model where real and nominal yields result from the interaction between infinitely risk-averse investors, who demand IL bonds with specific maturity, and risk-averse arbitrageurs, who allocate their real wealth in IL and nominal bonds based on a mean-variance problem. We begin by describing each of these investors, and then turn to the bond markets equilibria.

3.3.1 Financial structure

Let $P_{t,\tau}$ and $P^{\$}_{t,\tau}$ be the price at time t of an IL and a nominal zero-coupon bond, respectively, with a defined term to be redeemed $\tau \in (0, T]$. At maturity, IL bonds pay 1 unit of real wealth whereas nominal bonds pay 1 unit of currency. In other words, if an IL bond is held until maturity it will offer a real riskless return, free of real interest rate risk and inflation risk. On the other hand, if a nominal bond is carried to expiration it will yield a nominal riskless return, free of real interest rate risk but expose to inflation risk. The real and nominal yield are related to bonds' prices through

$$y_{t,\tau} = -\frac{\log[P_{t,\tau}]}{\tau}, \tag{3.1a}$$

$$y^{\$}_{t,\tau} = -\frac{\log[P^{\$}_{t,\tau}]}{\tau}. \tag{3.1b}$$

We assume that the instantaneous real riskless interest rate, r_t, and the instantaneous expected rate of inflation, π_t, follow two correlated Ornstein-Uhlenbeck processes:

$$dr_t = \kappa_r[\bar{r} - r_t]dt + \sigma_r dZ_{r,t}, \tag{3.2a}$$

$$d\pi_t = \kappa_\pi[\bar{\pi} - \pi_t]dt + \sigma_\pi dZ_{\pi,t}, \tag{3.2b}$$

where (κ_r, κ_π) are the mean-reverting parameters toward the unconditional long-run mean $(\bar{r}, \bar{\pi})$; the parameters (σ_r, σ_π) represent the volatility or diffusion coefficients; and $(Z_{r,t}, Z_{\pi,t})$ are two correlated standard Brownian motions with $dZ_{r,t}dZ_{\pi,t} = \rho_{r,\pi}dt$.

We assume there are two types of agents: PHI and arbitrageurs. In Vayanos and Vila (2009), PHI are assumed to consume at the end of their life, and to be infinitely risk-averse whereas arbitrageurs choose a bond portfolio to trade off instantaneous mean and variance. Their preferred-habitat framework does not distinguish between real and nominal terms and it does not incorporate the effect of inflation on agents' decisions. We introduce inflation into the model and assume that agents maximize over real consumption and wealth, respectively. Thus, in our model, PHI only invest in a IL bond with a maturity that matches their remaining life time[14] while arbitrageurs allocate their real wealth into IL and nominal bonds with several maturities.

[14]See for example Brennan and Xia (2002), Wachter (2003), and Cartea et al. (2012).

We use the term arbitrageurs to represent instantaneous mark-to-market investors that participate in the market of both class of bonds with different maturities to make profits. Hence, arbitrageurs avoid extreme segmentation in the IL market, integrating not only IL bond market across different maturities but the IL with the nominal bond market as well. We assume that arbitrageurs select a bond portfolio to trade off instantaneous mean and variance of the return of their real wealth. The arbitrageurs' optimization problem is

$$\max_{\{x_{t,\tau};x^{\$}_{t,\tau}\}_{\tau\in(0,T]}} \quad \mathrm{E}_t\left[\frac{dW_t}{W_t}\right]-\frac{\gamma}{2}\mathrm{Var}_t\left[\frac{dW_t}{W_t}\right] \tag{3.3}$$

subject to their budget constraint

$$\begin{aligned}\frac{dW_t}{W_t} =& \left[1-\int_0^T\left[x_{t,\tau}+x^{\$}_{t,\tau}\right]d\tau\right]r_t dt \\ &+\int_0^T\left[x_{t,\tau}\frac{dP_{t,\tau}}{P_{t,\tau}}+x^{\$}_{t,\tau}\frac{dP^{\$}_{t,\tau}}{P^{\$}_{t,\tau}}\right]d\tau \\ &-\left[\int_0^T x^{\$}_{t,\tau}d\tau\right]\pi_t dt.\end{aligned} \tag{3.4}$$

Here W_t represents the level of real wealth at time t; γ is a risk aversion coefficient; $x_{t,\tau}$ and $x^{\$}_{t,\tau}$ are the proportion of real wealth invested at t in the IL and nominal bond with maturity τ, respectively. The budget constraint in equation (3.4) represents the arbitrageurs' real wealth return that is equal to: (i) the return of the proportion of wealth invested in the instantaneous real riskless interest rate; (ii) the return of the proportion of wealth invested in IL bonds and nominal bonds; and, (iii) the loss for the proportion of wealth invested in nominal bonds because of inflation.

In our model, PHI constitute maturity clienteles for an IL bond with maturity τ representing the demand for buy-and-hold investors for these types of bonds. We assume that their demand relative to the arbitrageurs' level of real wealth ($D_{t,\tau}/W_t$) is an increasing and linear function of the real yield of the bond[15]

$$d_{t,\tau}=D_{t,\tau}/W_t=\tau\alpha(\tau)\left[y_{t,\tau}-\beta_{t,\tau}\right]. \tag{3.5}$$

Here $\beta_{t,\tau}$ represents the real return of alternative investment opportunities for PHI and α is a non-negative function in τ. Since PHI are assumed to be infinitely risk-averse and not affected by money illusion they do not substitute either across IL maturities or between IL and nominal bonds. In other words, PHI neither trade IL bonds that differ from their preferred-habitat maturities nor buy or sell nominal bonds. However, they are willing to save for future consumption in an alternative mean that yields an instantaneously real

[15] In Vayanos and Vila (2009) and Kaminska et al. (2011) preferred-habitat investors constitute maturity clienteles for a specific bond with maturity τ where the total demand (in absolute values) is an increasing affine function on the yield. In Hamilton and Wu (2011) the demand of preferred-habitat investors for nominal bonds is expressed relative to arbitrageurs wealth as well.

return $\beta_{t,\tau}$. For the time being we assume that $\alpha(\tau) = \alpha$ and $\beta_{t,\tau} = \beta$.

3.3.2 Bonds' market-clearing conditions

In equilibrium, the total demand of PHI and arbitrageurs must be equal to the total outstanding debt for each class of bonds, which is considered an exogenous variable in the model. The total outstanding debt can be understood as the net supply from other bond buyers not included in the model. For instance, central banks may affect the aggregate level of outstanding debt by buying or selling government securities through open market operations. Private foreign entities and non-private foreign entities may alter the total net supply of government debt as well.

Let the market-clearing conditions be

$$s_{t,\tau} = x_{t,\tau} + d_{t,\tau}, \tag{3.6a}$$
$$s^{\$}_{t,\tau} = x^{\$}_{t,\tau}, \tag{3.6b}$$

where $s_{t,\tau} = S_{t,\tau}/W_t$ and $s^{\$}_{t,\tau} = S^{\$}_{t,\tau}/W_t$ represent the total outstanding debt of IL, $S_{t,\tau}$, and nominal bond, $S^{\$}_{t,\tau}$, with maturity τ relative to the arbitrageurs' level of real wealth, respectively. Equation (3.6a) implies that the total outstanding debt of IL bond with maturity τ, must be equal to the sum of the demand of PHI and arbitrageurs. In the absence of arbitrageurs in the IL bond market, the term structure of real yields will exhibit extreme segmentation. the real yield for each maturity will be determined by the supply of IL bonds and the real return of alternative investment opportunities, $y^*_{t,\tau} = f(s_\tau, \beta)$. Equation (3.6b) implies that the total outstanding debt of nominal bond with maturity τ, must be absorbed by arbitrageurs since the assumption that PHI are infinitely risk-averse exclude them from the nominal bond market.[16]

For simplicity, we assume that the government adjusts the total outstanding debt to changes in the arbitrageurs' level of real wealth; thus s_τ and $s^{\$}_\tau$ are constant. This would be consistent with a debt management policy that tries to take advantage of arbitrageurs' level of real wealth. In Vayanos and Vila (2009) zero-coupon bonds are assumed to be in zero net supply, $s_{t,\tau} = 0$. Thus, in the absence of arbitrageurs the term structure of interest rate is flat at β and disconnected from the time-varying short rate. We consider a more general case where IL and nominal bonds can be in negative, zero, or positive net supply. However, the most interesting case is when bonds are in positive net supply and they can be bought by two class of investors: mark-to-market (arbitrageurs) and buy-and-hold (PH) investors.

[16] See Brennan and Xia (2002), and Wachter (2003).

3.3.3 Bond markets equilibria

Following the literature of affine models, we conjecture that in equilibrium the spot real yields of IL bonds are affine in r_t while the spot yields of nominal bonds are affine in r_t and π_t. Thus, bond prices of IL and nominal bonds are

$$P_{t,\tau} = e^{-\left[A(\tau)+B(\tau)r_t\right]}, \tag{3.7a}$$

$$P^{\$}_{t,\tau} = e^{-\left[A_{\$}(\tau)+B_{\$}(\tau)r_t+C_{\$}(\tau)\pi_t\right]}, \tag{3.7b}$$

where A, $A_{\$}$, B, $B_{\$}$ and $C_{\$}$ are functions that depend on τ but not on time t. Applying Ito's Lemma to (3.7a) and (3.7b) we get the instantaneous return of IL and nominal bonds

$$\frac{dP_{t,\tau}}{P_{t,\tau}} = \mu_{t,\tau}dt - B(\tau)\sigma_r dZ_{r,t}, \tag{3.8a}$$

$$\frac{dP^{\$}_{t,\tau}}{P^{\$}_{t,\tau}} = \mu^{\$}_{t,\tau}dt - B_{\$}(\tau)\sigma_r dZ_{r,t} - C_{\$}(\tau)\sigma_\pi dZ_{\pi,t}, \tag{3.8b}$$

where the drift terms are

$$\mu_{t,\tau} \equiv A'(\tau) + B'(\tau)r_t - B(\tau)\kappa_r\left[\bar{r} - r_t\right] + \frac{1}{2}B(\tau)^2\sigma_r^2, \tag{3.9a}$$

$$\begin{aligned}\mu^{\$}_{t,\tau} \equiv\ & A'_{\$}(\tau) + B'_{\$}(\tau)r_t + C'_{\$}(\tau)\pi_t - B_{\$}(\tau)\kappa_r\left[\bar{r} - r_t\right] - C_{\$}(\tau)\kappa_\pi\left[\bar{\pi} - \pi_t\right] \\ & + \frac{1}{2}\left[B_{\$}(\tau)^2\sigma_r^2 + C_{\$}(\tau)^2\sigma_\pi^2 + 2\rho_{r,\pi}\sigma_r\sigma_\pi B_{\$}(\tau)C_{\$}(\tau)\right].\end{aligned} \tag{3.9b}$$

Using (3.8)-(3.9) we can solve the arbitrageurs optimization problem stated in (3.3) subject to (3.4).

Lemma 3.3.1 (Arbitrageurs' first order conditions)**.** *The first order conditions with respect to the proportion invested in IL bonds, $x_{t,\tau}$, and with respect to the proportion invested in nominal bonds, $x^{\$}_{t,\tau}$ are given by*

$$\mu_{t,\tau} - r_t = B(\tau)\lambda_{r,t}, \tag{3.10a}$$

$$\mu^{\$}_{t,\tau} - \pi_t - r_t = B_{\$}(\tau)\lambda_{r,t} + C_{\$}(\tau)\lambda_{\pi,t}, \tag{3.10b}$$

where

$$\lambda_{r,t} \equiv \gamma\sigma_r^2 \int_0^T \left[x_{t,\tau}B(\tau) + x^{\$}_{t,\tau}B_{\$}(\tau)\right]d\tau + \gamma\rho_{r,\pi}\sigma_r\sigma_\pi \int_0^T x^{\$}_{t,\tau}C_{\$}(\tau)d\tau, \tag{3.11a}$$

$$\lambda_{\pi,t} \equiv \gamma\sigma_\pi^2 \int_0^T x^{\$}_{t,\tau}C_{\$}(\tau)d\tau + \gamma\rho_{r,\pi}\sigma_r\sigma_\pi \int_0^T \left[x_{t,\tau}B(\tau) + x^{\$}_{t,\tau}B_{\$}(\tau)\right]d\tau. \tag{3.11b}$$

In Lemma 3.3.1 $\lambda_{r,t}$ and $\lambda_{\pi,t}$ represent the market price of real interest rate risk and the market price of inflation risk, respectively. The sensitivity of IL bonds' returns with maturity τ to the real interest rate risk is given by $B(\tau)$ while the sensitivity of nominal bonds' returns with maturity τ to the short rate risk and inflation risk are $B_\$(\tau)$ and $C_\$(\tau)$, respectively. Arbitrageurs' first order conditions, equations (3.10), state that bond's expected real excess returns is proportional to the bond's sensitivity to each source of risk.

In (3.11a) the market price of real interest rate risk is given by the total exposure of arbitrageurs' portfolio to the instantaneous real riskless interest rate risk: (i) a direct exposure given by the holding of IL and nominal bonds; and, (ii) a cross-effect given by the correlation between the real interest rate and the inflation rate for arbitrageurs' holdings of nominal bonds.

Similarly, in (3.11b) the market price of inflation risk is determined by the total exposure of arbitrageurs' portfolio to inflation risk: (i) a direct exposure given by the holding of nominal bonds; and, (ii) a cross-effect given by the correlation between the inflation rate and the real interest rate for arbitrageurs' holdings of IL and nominal bonds. When $\rho_{r,\pi} \neq 0$ cross-effects on market prices of risk are zero.

3.4 Term Structure of Real Yields, Nominal Yields and BEI

In this section we determine the theoretical term structure of real and nominal yields predicted by the model in order to get implications about the BEI. We assume that demand factors are constant and that the government adjusts the total outstanding debt of nominal and IL bonds to changes in the arbitrageurs' level of real wealth.

3.4.1 Real yields

Proposition 3.4.1 expresses the real yield of IL bonds as an affine function of the instantaneous real riskless interest rate where $(\bar{r}^*, \kappa_r^*)$ are the long-run mean and the mean-reversion rate of the instantaneous real riskless rate under the risk-neutral measure;[17] and, Q_r represents the total outstanding debt's sensitivity to the real short rate risk. The solution of the standard Vasicek (1977) model is similar to equations (3.12a) and (3.12b) but the mean-reversion rate and the long-run mean under the risk-neutral measure are given by κ_r and $\tilde{r} = \bar{r} - \frac{\sigma_r}{\kappa_r}\lambda_r^{(Vasicek)}$, respectively.[18]

Proposition 3.4.1 (Yield of an IL bond)**.** *The real yield of an IL bond is an affine function of the instantaneous real riskless interest rate,* $y_{t,\tau} = \frac{1}{\tau}\Big[A(\tau) + B(\tau)r_t\Big]$*, where*

[17]See Appendix A.2.2.

[18]$\lambda_r^{(Vasicek)}$ represents the market price of risk that corresponds to the no-arbitrage condition in the Vasicek (1977) model.

the functions $A(\tau)$ *and* $B(\tau)$ *are given by*

$$A(\tau) = \kappa_r^* \bar{r}^* \int_0^\tau B(u)du - \frac{\sigma_r^2}{2} \int_0^\tau B(u)^2 du, \tag{3.12a}$$

$$B(\tau) = \frac{1 - e^{-\kappa_r^* \tau}}{\kappa_r^*}, \tag{3.12b}$$

where

$$\kappa_r^* \equiv \kappa_r + \gamma \sigma_r^2 \alpha \int_0^T B(\tau)^2 d\tau, \tag{3.13a}$$

$$\bar{r}^* \equiv \bar{r} - \frac{\sigma_r}{\kappa_r^*} \left[\frac{\left[\bar{r} - \beta\right] \gamma \sigma_r \alpha \int_0^T \tau B(\tau) d\tau - \frac{\gamma \sigma_r^3 \alpha}{2} \int_0^T B(\tau) \left[\int_0^\tau B(u)^2 du \right] d\tau - \gamma Q_r}{1 + \gamma \sigma_r^2 \alpha \int_0^T B(\tau) \left[\int_0^\tau B(u) du \right] d\tau} \right], \tag{3.13b}$$

$$Q_r \equiv \sigma_r \int_0^T \left[s_\tau B(\tau) + s_\tau^\$ B_\$(\tau) \right] d\tau + \rho_{r,\pi} \sigma_\pi \int_0^T s_\tau^\$ C_\$(\tau) d\tau. \tag{3.13c}$$

In the absence of PHI, $\alpha = 0$, the model is equivalent to the standard Vasicek (1977) where the risk is determined by the arbitrageurs' risk aversion coefficient and the total outstanding level of IL and nominal debt relative to the arbitrageurs' level of real wealth. The mean-reversion rate of the instantaneous real riskless rate under the risk-neutral measure is equal to the parameter under the true measure ($\kappa_r^* = \kappa_r$) and the long-run mean of the instantaneous real riskless rate under the risk-neutral measure is

$$\bar{r}^* = \tilde{r} = \bar{r} + \frac{\sigma_r}{\kappa_r} \underbrace{\gamma \left[\sigma_r \int_0^T \left[s_\tau B(\tau) + s_\tau^\$ B_\$(\tau) \right] d\tau + \rho_{r,\pi} \sigma_\pi \int_0^T s_\tau^\$ C_\$(\tau) d\tau \right]}_{-\lambda_r^{(Vasicek)}}. \tag{3.14}$$

For simplicity assume $\rho_{r,\pi} = 0$ and $\gamma > 0$, then (3.14) states that the long-run mean under the risk-neutral measure will depend positively on the total outstanding level of IL and nominal debt relative to the arbitrageurs' level of real wealth.

When arbitrageurs are risk-averse, the mean-reversion rate under the risk-neutral measure is affected by PHI and the short rate reverts faster to the long-run mean under the risk-neutral than under the true measure ($\kappa_r^* > \kappa_r$). This result allow Vayanos and Vila (2009) to show the positive slope-premiums relationship stated by Fama and Bliss (1987), and the under-reaction of forward rates to changes in expected spot rates. Particularly, they show bond risk premiums reflect arbitrageurs' carry roll-up (roll-down) trades when the short rate is high (low), arbitrageurs are short (long) long-term bonds. The long-run mean under the risk-neutral measure differs from its true counterpart as well as in the Vasicek model. Re-expressing (3.13b) as

$$\bar{r}^* - \bar{r} = c_0 + c_1 \left[\beta - \bar{r}\right] + c_2 \sigma_r Q_r, \tag{3.15}$$

where

$$c_0 = c_2 \frac{\sigma_r^4}{2} \alpha \int_0^T B(\tau) \Big[\int_0^\tau B(u)^2 du \Big] d\tau \geq 0, \tag{3.16a}$$

$$c_1 = c_2 \sigma_r^2 \alpha \int_0^T \tau B(\tau) d\tau \geq 0, \tag{3.16b}$$

$$c_2 = \frac{\gamma}{c_3} \geq 0, \tag{3.16c}$$

$$c_3 = \kappa_r^* \Big[1 + \gamma \sigma_r^2 \alpha \int_0^T B(\tau) \int_0^\tau B(u) du \, d\tau \Big] > 0, \tag{3.16d}$$

we can disentangle the effect of PHI and the total outstanding debt over the long-run mean under the risk-neutral measure. The first two terms (c_0, c_1) appear as a consequence of PHI; while the last one is a term proportional to the total outstanding debt sensitivity to the real short rate risk. Interestingly, the long-run mean under the risk-neutral measure is affected by the real return of alternative investment opportunities and by the total outstanding level of IL and nominal debt relative to the arbitrageurs' level of real wealth.

In Figure 3.6 the solid (black) line corresponds to the real yield curve for the standard Vasicek case where the price of risk is determined by the arbitrageurs' risk aversion coefficient and the total outstanding level of IL and nominal debt relative to the arbitrageurs' level of real wealth. The dash (blue) line represent the effect of PHI over the real yield curve. The Figure shows a flattest yield curve as compared with the standard Vasicek model. The presence of PHI reduces the amount of debt that must be absorbed by arbitrageurs affecting the market price of real rate risk.

Figure 3.6a shows the effect of a change in the real return of alternative investment opportunities for PHI. An increase in β reduces the demand of PHI which leads arbitrageurs to increase their holdings of IL debt affecting the market price of real rate risk. The opposite result is obtained after a decrease in the real return of alternative investment opportunities. Figure 3.6b exhibits the effect of the correlation between the real interest rate and the inflation rate over the real yield curve. When arbitrageurs' bond portfolio is composed by both types of debt, the portfolio's risk is augmented when this correlation is positive and reduced when the correlation is negative.

3.4.2 Nominal yields

Proposition 3.4.2 (see Appendix A.2.2) expresses the yield of nominal bonds as an affine function of the instantaneous real riskless interest rate and the expected rate of inflation. When arbitrageurs are risk-neutral ($\kappa_r^* = \kappa_r, \delta = 0, \bar{r}^* = \bar{r}, \bar{\pi}^* = \bar{\pi}$), the affine coefficients presented in Proposition 3.4.2 are equal to the coefficient stated by the two-factor Vacicek model by Chen (1995).

Proposition 3.4.2 (Yield of a nominal bond)**.** *The yield of a nominal bond is an affine function of the instantaneous real riskless interest rate and the instantaneous expected rate* *of inflation,* $y^{\$}_{t,\tau} = \frac{1}{\tau}\Big[A_{\$}(\tau) + B_{\$}(\tau)r_t + C_{\$}(\tau)\pi_t\Big]$*, where functions* $A_{\$}(\tau)$*,* $B_{\$}(\tau)$ *and* $C_{\$}(\tau)$ *are given by*

$$\begin{aligned} A_{\$}(\tau) = {} & \kappa_r^* \bar{r}^* \int_0^\tau B_{\$}(u)du + \kappa_\pi \bar{\pi}^* \int_0^\tau C_{\$}(u)du \\ & - \frac{1}{2}\Big[\sigma_r^2 \int_0^\tau B_{\$}(u)^2 du + \sigma_\pi^2 \int_0^\tau C_{\$}(u)^2 du + 2\rho_{r,\pi}\sigma_r\sigma_\pi \int_0^\tau B_{\$}(u)C_{\$}(u)du\Big], \end{aligned} \tag{3.17a}$$

$$B_{\$}(\tau) = \frac{1 - e^{-\kappa_r^*\tau}}{\kappa_r^*} + \delta\Big[\frac{1 - e^{-\kappa_\pi\tau}}{\kappa_\pi} - \frac{1 - e^{-\kappa_r^*\tau}}{\kappa_r^*}\Big], \tag{3.17b}$$

$$C_{\$}(\tau) = \frac{1 - e^{-\kappa_\pi\tau}}{\kappa_\pi}, \tag{3.17c}$$

where

$$\bar{\pi}^* \equiv \bar{\pi} + \frac{\sigma_\pi}{\kappa_\pi}\gamma\Big[\sigma_\pi N_\pi + \rho_{r,\pi}\sigma_r N_r\Big], \tag{3.18a}$$

$$\delta \equiv \frac{\gamma\rho_{r,\pi}\sigma_r\sigma_\pi\alpha\int_0^T B(\tau)^2 d\tau}{\kappa_\pi - \kappa_r^*} = -\rho_{r,\pi}\frac{\sigma_\pi}{\sigma_r}\frac{[\kappa_r^* - \kappa_r]}{[\kappa_r^* - \kappa_\pi]}, \tag{3.18b}$$

$$N_r \equiv \int_0^T \alpha B(\tau)\Big[\tau\beta - A(\tau)\Big]d\tau + \int_0^T \Big[s_\tau B(\tau) + s_\tau^{\$} B_{\$}(\tau)\Big]d\tau, \tag{3.18c}$$

$$N_\pi \equiv \int_0^T s_\tau^{\$} C_{\$}(\tau)d\tau. \tag{3.18d}$$

When arbitrageurs are risk-averse and there are no PHI in the IL bond market, it can be shown that

$$B_{\$}(\tau) = B(\tau) = \frac{1 - e^{-\kappa_r\tau}}{\kappa_r}, \tag{3.19}$$

and

$$\bar{\pi}^* = \tilde{\pi} = \bar{\pi} + \frac{\sigma_\pi}{\kappa_\pi}\underbrace{\gamma\Big[\sigma_\pi\int_0^T s_\tau^{\$} C_{\$}(\tau)d\tau + \rho_{r,\pi}\sigma_r\int_0^T\Big[s_\tau B(\tau) + s_\tau^{\$} B_{\$}(\tau)\Big]d\tau\Big]}_{-\lambda_\pi^{(Vasicek)}}. \tag{3.20}$$

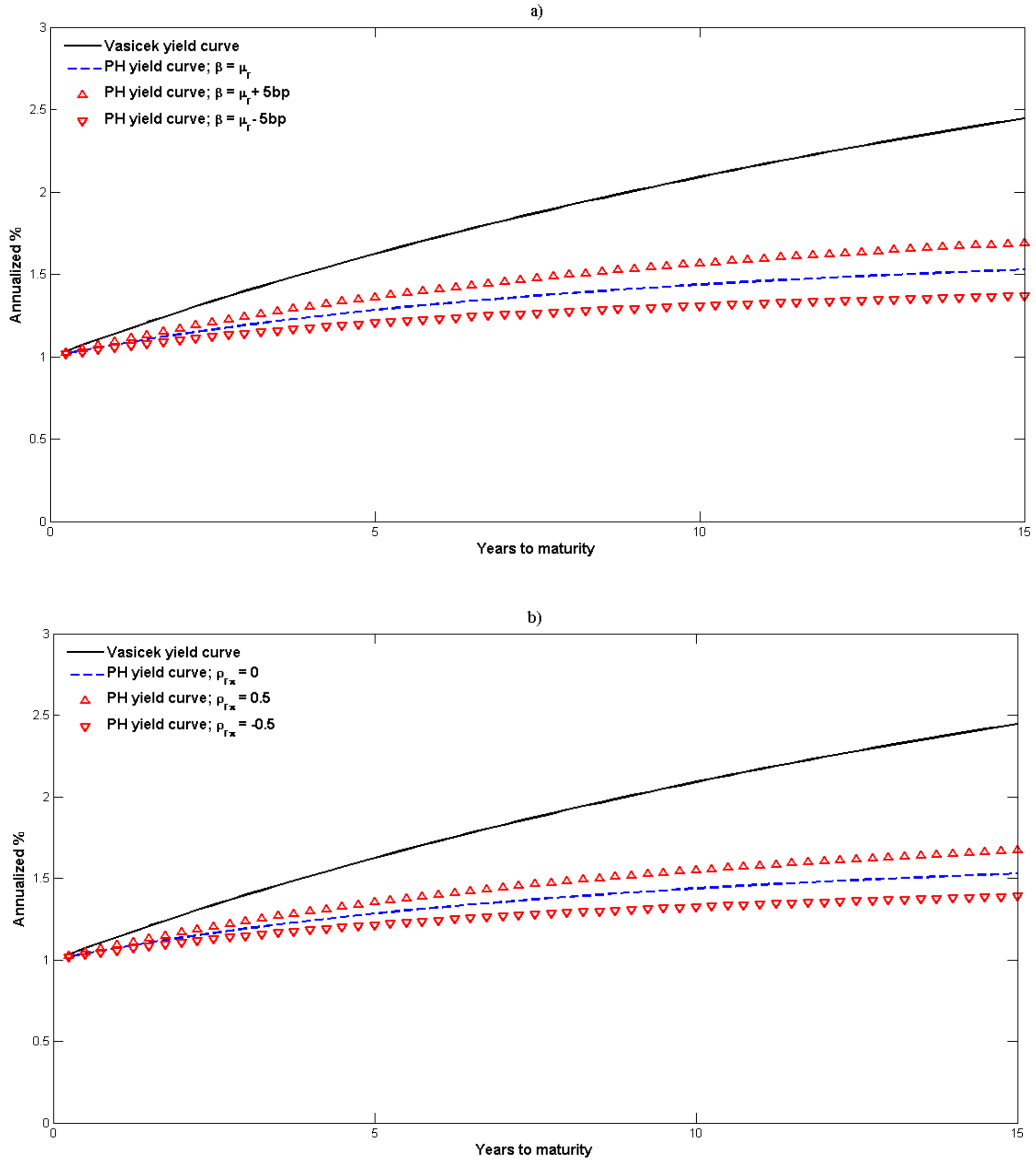
The mean-reversion rates under the risk-neutral measure (κ_r, κ_π) are equal to their true counterparts. This result implies that the sensitivity of nominal yields to the real short rate is equal to the sensitivity of real yields to the real short rate, $B_{\$}(\tau) = B(\tau)$. The long-run mean of $\bar{r}^*$ and $\bar{\pi}^*$ will depend on the arbitrageurs' risk aversion coefficient and the total outstanding level of IL and nominal debt relative to the arbitrageurs' level of real wealth. Arbitrageurs will consider the risk of adding IL or nominal bonds into their portfolio which results that in equilibrium long-run mean under the risk-neutral measure is higher than their true counterparts. Thus, when there are only arbitrageurs in the bond markets the solution of the model is equivalent to the two-factor Vasicek model where market prices of risk depend on the arbitrageurs' risk aversion coefficient and the total outstanding level of IL and nominal debt relative to the arbitrageurs' level of real wealth.

When arbitrageurs are risk-averse and PHI participate in the IL bond market, the sensitivity of nominal yields to the real short rate may differ from the sensitivity of real yields to the real short rate. In particular, when the correlation between the real riskless rate and the inflation rate is positive, real yields are more sensitive than nominal yields to the real short rate, $B(\tau) > B_{\$}(\tau)$. On the contrary, real yields are less sensitive than nominal yields to the real short rate when the correlation between the real riskless rate and the inflation rate is negative, $B(\tau) < B_{\$}(\tau)$. Finally, if the real riskless rate is uncorrelated with inflation ($\rho_{r,\pi} = 0 \Rightarrow \delta = 0$), nominal and real yields with equal term to mature will have the same sensitivity to the instantaneous real riskless rate, $B(\tau) = B_{\$}(\tau)$.

In Figure 3.7 the solid (black) line corresponds to the nominal yield curve for the 2-factor Vasicek case where the prices of risk are determined by the arbitrageurs' risk aversion coefficient and the total outstanding level of IL and nominal debt relative to the arbitrageurs' level of real wealth. The dash (blue) line represent the effect of PHI over the nominal yield curve. The Figure shows a flattest yield curve as compared with the 2-factor Vasicek model. The presence of PHI reduces the amount of IL debt that must be absorbed by arbitrageurs affecting the market price of real rate risk.

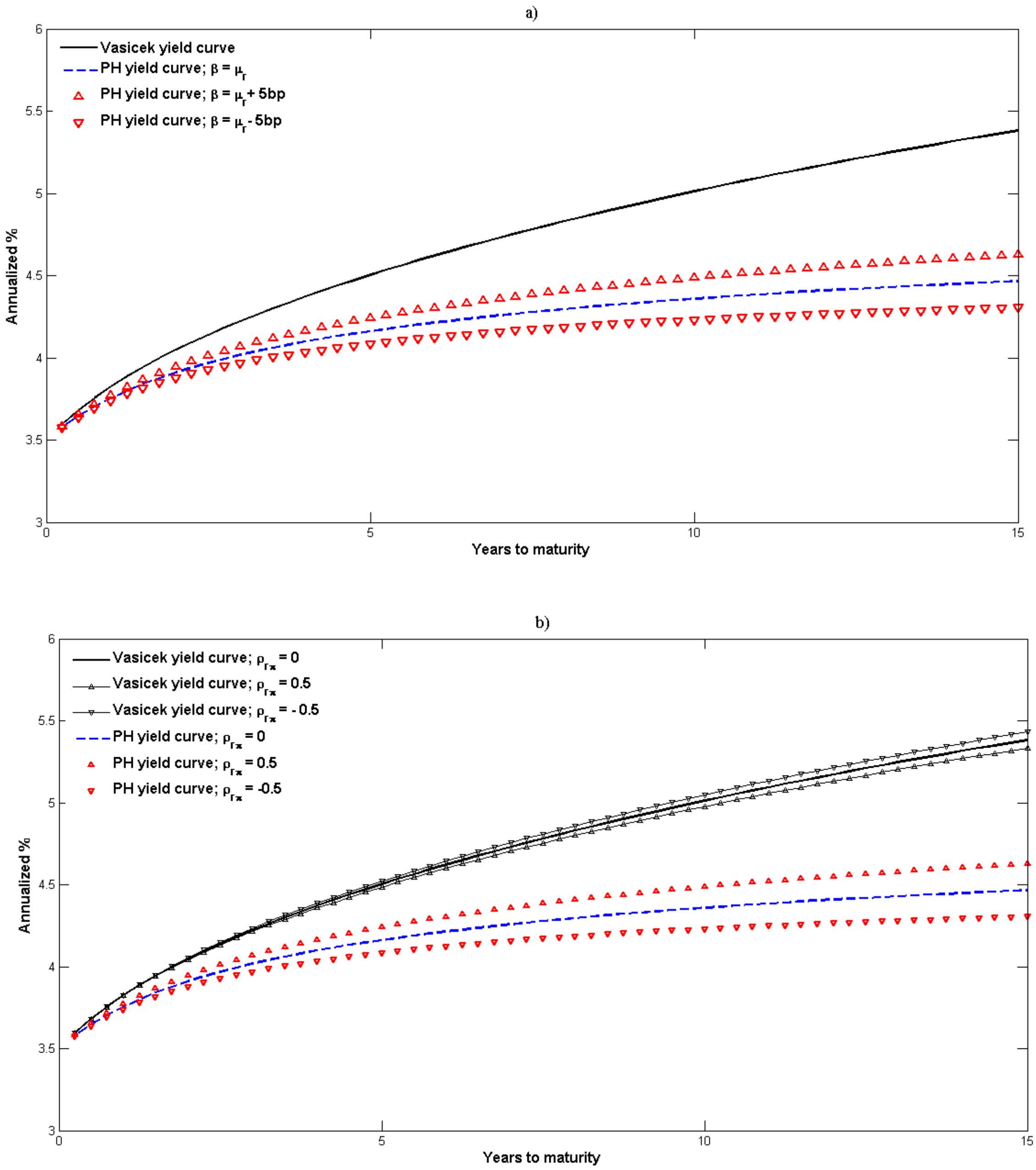
Figure 3.7a shows the effect of a change in the real return of alternative investment opportunities over the nominal yield curve. In the 2-factor Vasicek case there is no link between bond prices and the real return of alternative investment opportunities. In the preferred-habitat model an increase in β reduces the demand of PHI which leads arbitrageurs to increase their holdings of IL debt affecting the market price of real rate risk. The opposite result is obtained after a decrease in the real return of alternative investment opportunities.

Figure 3.7b exhibits the effect of the correlation between the real rate and the inflation rate over the nominal yield curve. In the 2-factor Vasicek case a change in the correlation will affect $A_{\$}(\tau)$ just through the convexity term. In the case of the preferred-habitat model the correlation coefficient plays a crucial role since it affects the total exposition of arbitrageurs' to the two risk factors. Thus, the correlation coefficient affects $A_{\$}(\tau)$ through the convexity term and through the market prices of risk. When arbitrageurs' bond portfolio is composed by both types of debt, the portfolio's risk increases when this correlation is positive and decreases when the correlation is negative.



a) Effect of a change in the real return of alternative investment opportunities for PHI. The solid (black) line corresponds to the Vasicek case; we assume $\kappa_r = 0.06$, $\mu_r = 0.01$, $\sigma_r = 0.01$, and $\lambda_r = -0.29$. Specifying also the parameters governing the instantaneous inflation rate in equation (3.2), we obtain the total outstanding debt for each class of bonds which results in $S/S^{\$} = 12\%$, see Appendix C.1. The dashed (blue) line refers to the real yield curve when there are PHI; we assume $\beta = \mu_r$, $\alpha = 1$, $\gamma = 2$, $\rho_{r\pi} = 0$. The upward (downward) pointing marker corresponds to an increase (decrease) in 5bp in the real return of alternative investment opportunities. **b)** Effect of a change in the correlation between the real rate and the inflation. The solid (black) line correspond to the Vasicek case. The dashed (blue) line refers to the real yield curve when there are PHI. The upward (downward) pointing marker corresponds to a change in the correlation coefficient from $\rho_{r\pi} = 0$ to 0.5 (-0.5).

Figure 3.6: Real yield curve under the preferred-habitat model



a) Effect of a change in the real return of alternative investment opportunities. The solid (black) line correspond to the 2-Factor Vasicek case; we assume $\kappa_r = 0.06$, $\mu_r = 0.01$, $\sigma_r = 0.01$, $\lambda_r = -0.29$, and $\kappa_\pi = 0.995$, $\mu_\pi = 0.025$, $\sigma_\pi = 0.035$, and $\lambda_\pi = -0.15$. Then, we obtain the total outstanding debt for each class of bonds which results in $S/S^{\$} = 12\%$, see Appendix C.1. The dashed (blue) line refers to the nominal yield curve when there are PHI; we assume $\beta = \mu_r$, $\alpha = 1$, $\gamma = 2$, $\rho_{r\pi} = 0$. The upward (downward) pointing marker corresponds to an increase (decrease) in 5bp in the real return of alternative investment opportunities. **b)** Effect of a change in the correlation between the real rate and the inflation rate. The solid (black) line correspond to the 2-Factor Vasicek case. The dashed (blue) line refers to the nominal yield curve when there are PHI. The upward (downward) pointing marker corresponds to a change in the correlation coefficient from $\rho_{r\pi} = 0$ to 0.5 (-0.5) for each case.

Figure 3.7: Nominal yield curve under the preferred-habitat model

3.4.3 Breakeven inflation rates

The BEI is the difference in yields between a nominal bond and an IL bond with the same maturity

$$b_{t,\tau} \equiv y^{\$}_{t,\tau} - y_{t,\tau}. \tag{3.21}$$

Ex-post, the BEI represents the average realized inflation rate over τ that equates the buy-and-hold return of both type of bonds. Consider the buy-and-hold self-financing strategy where an investor buys an IL bond and simultaneously sells a nominal bond with the same maturity. If realized inflation over τ is higher than $b_{t,\tau}$ then the IL bond will outperform the nominal bond and the investor will make money. On the contrary, if realized inflation over τ is lower than $b_{t,\tau}$ then the IL bond will underperform the nominal bond and the strategy will lose money.

Ex-ante, it is argued that in a risk-neutral world the BEI with maturity τ should be equal to the average expected inflation rate over τ, which is the well known PEH or Fisher's hypothesis. In a risk-averse world, buy-and-hold investors will be willing to pay a positive premium to buy IL instead of nominal bonds for the protection they will receive against inflation risk. Then, it is said the BEI should be equal to the average expected inflation rate plus a positive inflation risk premium. Using Propositions 3.4.1 and 3.4.2 we express the BEI in Corollary 3.4.3.

Corollary 3.4.3 (Breakeven inflation rates)**.** *The breakeven inflation rate is an affine function of the instantaneous real riskless interest rate and the instantaneous expected rate of inflation,* $b_{t,\tau} = \frac{1}{\tau}\Big[A_b(\tau) + B_b(\tau)r_t + C_b\pi_t\Big]$, *where*

$$\begin{aligned} A_b(\tau) =& \kappa_r^*\bar{r}^* \int_0^\tau \Big[B_\$(u) - B(u)\Big]du + \kappa_\pi\bar{\pi}^* \int_0^\tau C_\$(u)du - \rho_{r,\pi}\sigma_r\sigma_\pi \int_0^\tau B_\$(u)C_\$(u)du \\ & - \frac{1}{2}\Big[\sigma_r^2 \int_0^\tau \Big[B_\$(u)^2 - B(u)^2\Big]du + \sigma_\pi^2 \int_0^\tau C_\$(u)^2 du\Big], \end{aligned} \tag{3.22a}$$

$$B_b(\tau) = \delta\Big[\frac{1-e^{-\kappa_\pi\tau}}{\kappa_\pi} - \frac{1-e^{-\kappa_r^*\tau}}{\kappa_r^*}\Big] = \delta\Big[C_\$(\tau) - B(\tau)\Big], \tag{3.22b}$$

$$C_b(\tau) = C_\$(\tau) = \frac{1-e^{-\kappa_\pi\tau}}{\kappa_\pi}. \tag{3.22c}$$

Corollary 3.4.3 shows when $\delta \neq 0$, the BEI is an affine function of the expected inflation rate and of the real riskless interest rate as well. Remember from (3.18b) that δ might be zero when arbitrageurs are risk-neutral or when the inflation rate and the instantaneous real riskless interest rate are uncorrelated. When arbitrageurs are risk-neutral, the BEI with maturity τ can be written as function of two main components: the inflation long-run mean, and the current level of inflation with respect to the long-run mean,

$$b^{(\gamma=0)}_{t,\tau} = \bar{\pi} + \frac{C_\$(\tau)}{\tau}\big[\pi_t - \bar{\pi}\big] - O(\tau), \tag{3.23}$$

where $O(\tau)$ refers to a convexity term which is negligible compared to the other two terms. Equation (3.23) refers to the case of the pure expectations hypothesis (PEH) by Fisher (1896) and Lutz (1940) where the BEI term structure is driven by the investors' expectations on futures instantaneous inflation rates and all term premia are zero.

When arbitrageurs are risk-averse, the preferred-habitat model is the same as the two-factor Vasicek model where the term structure of BEI is given by

$$\begin{aligned} b_{t,\tau}^{(\gamma>0)} &= \tilde{\pi} + \frac{C_{\$}(\tau)}{\tau}\big[\pi_t - \tilde{\pi}\big] - O(\tau), \\ &= b_{t,\tau}^{(\gamma=0)} + \lambda_{\pi}\frac{\sigma_{\pi}}{\kappa_{\pi}}\left[1 - \frac{C_{\$}(\tau)}{\tau}\right] - O(\tau), \end{aligned} \tag{3.24}$$

where

$$\lambda_{\pi} = \gamma\Big[\sigma_{\pi}\int_0^T s_{\tau}^{\$}C_{\$}(\tau)d\tau + \rho_{r,\pi}\sigma_r\int_0^T \big[s_{\tau}B(\tau) + s_{\tau}^{\$}B_{\$}(\tau)\big]d\tau\Big]. \tag{3.25}$$

Equation (3.24) represents the case of the expectations hypothesis (EH). It differs from (3.23) since it states that the term structure of BEI is driven by the investors' expectations on futures instantaneous inflation rates plus a inflation risk premium that depends on maturity but not on time. The inflation risk premium is given by the difference between $\tilde{\pi}$ and $\bar{\pi}$. Given equation (3.18a) we can argue two implications for the BEI. First, the EH will hold over the term structure of BEI if the total outstanding debt of nominal and IL bonds relative to the arbitrageurs' level of real wealth is constant through time.[19] Second, that the term premium can be either positive or negative. Particularly, when the correlation between the inflation and the real interest rate is negative the term premium can be negative. The intuition is that when the main source of risk of arbitrageurs' portfolio is given by the real interest rate risk, the inflation uncertainty will behave as a hedger to the real rate risk.

Corollary 3.4.3 shows the term structure of the BEI in the general case when arbitrageurs are risk-averse and PHI participate in the IL bond market. When the inflation rate and the real short rate are uncorrelated, the BEI is given by equation (3.24). The main implication of Corollary 3.4.3 is that the inflation risk premium in the BEI will depend on the real short rate if the correlation between inflation and the real short rate is different from zero. Under these circumstances the EH will not hold in the term structure of BEI since the term premium will depend on the maturity of the bond and on time,

$$b_{t,\tau}^{(ph)} = b_{t,\tau}^{(\gamma=0)} + \lambda_{\pi}^{(ph)}\frac{\sigma_{\pi}}{\kappa_{\pi}}\left[1 - \frac{C_{\$}(\tau)}{\tau}\right] + d_0 + d_1 r_t - O(\tau), \tag{3.26}$$

[19] Pflueger and Viceira (2011) empirically reject the EH in the US and UK government BEI term structure.

where

$$\lambda_\pi^{(ph)} = \lambda_\pi + \gamma\sigma_r\rho_{r,\pi}\alpha\int_0^T B(\tau)\Big[\tau\beta - A(\tau)\Big]d\tau, \tag{3.27a}$$

$$d_0 = \delta\bar{r}^*\left[\frac{\kappa_r^*}{\kappa_\pi}\Big[1 - \frac{C_\$(\tau)}{\tau}\Big] - \Big[1 - \frac{B(\tau)}{\tau}\Big]\right], \tag{3.27b}$$

$$d_1 = \delta\frac{C_\$(\tau) - B(\tau)}{\tau}. \tag{3.27c}$$

The presence of PHI affects the BEI through the inflation risk premium. In equilibrium, the total demand of PHI and arbitrageurs must be equal to the total outstanding of IL bonds (equation 3.6a). A higher real short rate implies that PHI will increase their demand of IL bonds (equation 3.5 and Proposition 3.12) and arbitrageurs will reduce their holdings of IL bonds. In equation (3.11b), the market price of inflation risk is determined by the total exposure of arbitrageurs' portfolio to inflation risk which is given by (i) a direct exposure given by the holding of nominal bonds; and, (ii) a cross-effect given by the correlation between the inflation rate and the real rate for arbitrageurs' holdings of IL and nominal bonds. Then, if the correlation between the inflation rate and the real short rate is positive (negative) the BEI inflation risk premium will decrease (increase) after an increase in the real short rate.

3.4.4 Forward rates

A forward rate is an alternative useful way to summarize the information in the term structure of interest rates. The forward rate curve provides some benefits over the yield curve since the former summarizes information about future short rates in a way that makes it easier to interpret. A forward rate, $f_{t,\tau-\Delta\tau,\tau}$, represents the bond's rate of return from $\tau - \Delta\tau$ to τ committed at time t and can be obtained from bonds' prices through

$$f_{t,\tau-\Delta\tau,\tau} = -\frac{\log\big(P_{t,\tau}/P_{t,\tau-\Delta\tau}\big)}{\Delta\tau}. \tag{3.28}$$

In the limit when $\Delta\tau \to 0$ we get the instantaneous forward rate $f_{t,\tau} = -\frac{\partial\log(P_{t,\tau})}{\partial\tau}$, which can be interpreted as the marginal change in the total return from an infinitesimal increase in the length time of the bond.

The difference between the nominal forward rate and the real forward rate is described as the forward BEI, and it is usually interpreted as the compensation that investors demand both for expected inflation and for the risk associated with that inflation. In the model the instantaneous forward BEI is given by

$$f_{t,\tau}^{bei} \equiv \lim_{\Delta\tau\to 0} f_{t,\tau-\Delta\tau,\tau}^{bei} = -\frac{\partial\log\big(P_{t,\tau}^{\$}/P_{t,\tau}\big)}{\partial\tau} = A_b'(\tau) + B_b'(\tau)r_t + C_b'\pi_t, \tag{3.29}$$

implying that in the presence of PHI, the forward BEI might be also affected by the real riskless interest rate. From Corollary 3.4.3, $B_b'(\tau)$ is different from zero when arbitrageurs are risk-averse, and the inflation and the real rate are correlated.

Proposition 3.4.4 (Sensitivities of forward rates to real interest shocks)**.** *A unit shock to the real short rate r_t*

- *Raises the expected real short rate at $t+\tau$ by $\frac{\partial E_t(r_t)}{\partial r_t} = e^{-\kappa_r \tau}$.*
- *Raises the instantaneous real forward rate $f_{t,\tau}$ by $\frac{\partial f_{t,\tau}}{\partial r_t} = e^{-\kappa_r^* \tau}$.*
- *Changes the instantaneous nominal forward rate $f_{t,\tau}^{\$}$ by $\frac{\partial f_{t,\tau}^{\$}}{\partial r_t} = [1-\delta]e^{-\kappa_r^* \tau} + \delta e^{-\kappa_\pi \tau}$.*
- *Changes the instantaneous forward BEI $f_{t,\tau}^{bei}$ by $\frac{\partial f_{t,\tau}^{bei}}{\partial r_t} = \delta\big[e^{-\kappa_\pi \tau} - e^{-\kappa_r^* \tau}\big]$.*

Proposition 3.4.4 states the reaction of forward rates to changes in the real short rate. Vayanos and Vila (2009) show that in the presence of PHI, forward rates under-react to changes in expected spot rates. That is, risk-averse arbitrageurs partially incorporate information about expected short rate into forward rates and the degree of under-reaction depends on the level of risk aversion. Specifically, the more risk-averse are arbitrageurs the stronger is the under-reaction of forward real rates.

The intuition of the previous finding is as follows. A higher real short rate implies that PHI will increase their desired holdings of IL bonds leading arbitrageurs to reduce their holdings of IL bonds. Then, the market price of real rate risk will decrease and partially off-set the increase in the real short rate. The more risk-averse are arbitrageurs, the stronger is the effect over the market price of the real rate risk.

Proposition 3.4.4 shows that forward nominal rates might react differently from real forward rates to changes in the real short rate. When the inflation rate and the real short rate are correlated, there is a cross-effect of the real short rate over the market price of the inflation risk. Particularly, when the inflation rate and the real rate are positively correlated, long-term nominal bonds are riskier than long-term IL bonds for arbitrageurs, thus the under-reaction of nominal forward rates after a change in the real rate will be stronger. The reduction of arbitrageurs' holding of IL bonds leads to a reduction not only in the market price of the real rate risk but also in the market price of the inflation risk. On the contrary, if the inflation rate and the real rate are negatively correlated, then the under-reaction of forward nominal rates will be weaker since the market price of the inflation risk will increase after an increase of the real short rate.

This last results implies that when arbitrageurs risk aversion is high and the inflation rate and the real short rate are correlated, the under-reaction of forward real rates and forward nominal rates will differ most. In the extreme case when $\gamma \to \infty$, then $\frac{\partial f_{t,\tau}}{\partial r_t} = 0$ and $\frac{\partial f_{t,\tau}^{\$}}{\partial r_t} = -\rho_{r,\pi}\frac{\sigma_\pi}{\sigma_r}e^{-\kappa_\pi \tau}$. In other words, when arbitrageurs are extremely risk-averse forward real rates will not be altered by changes in the real short rate while forward nominal rates will do.

The instantaneous forward BEI will be affected by changes in the real rate when the inflation and the real rate are correlated. When inflation and the real rate are positively correlated, nominal yields under-reaction is stronger than real yields under-reaction leading a decrease in the BEI. In the other case, when inflation and the real rate are negatively correlated, an increase in the real rate will lead to an increase in the BEI.

Proposition 3.4.5 (Sensitivities of forward rates to inflation shocks)**.** *A unit shock to inflation rate* π_t

- *Raises the expected inflation rate at* $t + \tau$ *by* $\frac{\partial E_t(\pi_t)}{\partial \pi_t} = e^{-\kappa_\pi \tau}$.
- *Raises the instantaneous nominal forward rate* $f^{\$}_{t,\tau}$ *by* $\frac{\partial f^{\$}_{t,\tau}}{\partial \pi_t} = e^{-\kappa_\pi \tau}$.
- *Raises the instantaneous forward BEI* $f^{bei}_{t,\tau}$ *by* $\frac{\partial f^{bei}_{t,\tau}}{\partial \pi_t} = e^{-\kappa_\pi \tau}$.

Proposition 3.4.5 states the reaction of forward rates to changes in expected inflation. The presence of PHI in the IL bond market does not alter the impact of π_t on forward rates compared to the standard Vasicek model. Thus, the forward BEI incorporates a complete information about future expected inflation.

As a result, the preferred-habitat model states that when the risk aversion of arbitrageurs is high, the forward BEI may be a biased measure of the compensation that buy-and-hold investors demand to cover the expected rate of inflation. That is, the forward BEI will include the compensation that investors demand for expected inflation, the risk associated with that inflation, and a term proportional to the real short rate that affects the market price of inflation through the cross-effect given by the correlation coefficient between the inflation rate and the real rate.

3.4.5 Market prices of risk

Re-expressing the market price of real short rate risk, equation (3.11a); with the market-clearing conditions, equations (3.6a)-(3.6b); the demand of PHI, equation (3.5); and bonds' yields and prices, equations (3.1a)-(3.1b) and equations (3.7a)-(3.7b), we obtain

$$\lambda_{r,t} = \gamma \sigma_r \Big[- a_0 + a_1 \beta - a_2\, r_t + Q_r \Big] \tag{3.30}$$

where

$$a_0 \equiv \sigma_r \alpha \int_0^T A(\tau)B(\tau)\, d\tau \tag{3.31a}$$

$$a_1 \equiv \sigma_r \alpha \int_0^T \tau B(\tau)\, d\tau \geq 0 \tag{3.31b}$$

$$a_2 \equiv \sigma_r \alpha \int_0^T B(\tau)^2\, d\tau \geq 0 \tag{3.31c}$$

$$Q_r \equiv \sigma_r \int_0^T \Big[s_\tau B(\tau) + s_\tau^{\$} B_{\$}(\tau) \Big] d\tau + \rho_{r,\pi}\sigma_\pi \int_0^T s_\tau^{\$} C_{\$}(\tau) d\tau. \tag{3.31d}$$

Equation (3.30) states that the market price of real interest rate risk as a function of arbitrageurs' risk aversion coefficient, the real return of alternative investment opportunities, the total outstanding level of IL and nominal bonds relative to the arbitrageurs' level of real wealth, and the level and volatility of the real short rate. The real short rate risk premium can be interpreted as the total exposure of arbitrageurs' portfolio to changes in the instantaneous real riskless interest rate.[20]

In the absence of PHI, $\alpha = 0$, a_0, a_1 and a_2 vanish and the market price of real short rate risk depends on the arbitrageurs' risk aversion coefficient, the volatility of the real short rate and the total outstanding level of IL and nominal bonds relative to the arbitrageurs' level of real wealth. All these parameters are constant and the real short rate risk premium is constant across time.

The actions of PHI affect the market price of risk of the real short rate. Specifically, if the real yield is low relative to the real return of alternative investment opportunities, then PHI will be net sellers of IL bonds and arbitrageurs will be more exposed to the real short rate risk. On the contrary, when PHI are net buyers of IL bonds, the total exposure of arbitrageurs to changes in the real short rate will be lower than in the case without PHI. The result is that in equilibrium the real short rate risk premium is an affine and decreasing function of r_t which underlies the time-varying behavior of $\lambda_{r,t}$ and the rejection of the EH.

Proposition 3.4.6 (Market price of real interest rate risk and supply of IL bonds)**.** *The market price of real interest rate risk depends positively on the supply on IL bonds. An increase in the supply of a given maturity s_τ of IL bonds affects the real short rate premium*

[20] Lou et al. (2012) study the temporary price impacts of Treasury security auctions on the secondary Treasury and repo markets. They find that Treasury auctions exerts significant price pressure in the secondary Treasury market during days surrounding the auction process. They suggest that the phenomenon can be explained by the limited risk bearing capacity of primary dealers and limited mobility of end-investors' capital. They also find that the price impact is exacerbated when the total risk to be acquired by primary dealers is larger, that is when the auction size is larger or when interest rates are more volatile.

by

$$\frac{\partial \lambda_{r,t}}{\partial s_\tau} = \gamma\sigma_r^2 B(\tau) - \gamma\sigma_r^2 \alpha \int_0^T \frac{\partial A(\tau)}{\partial s_\tau} B(\tau)\, d\tau \geq 0. \tag{3.32}$$

Proposition 3.4.6 shows that an increase in the supply of IL bonds positively affects the market price of real interest rate risk and the magnitude of this effect depends on: (i) the degree of risk aversion of arbitrageurs and the real interest rate volatility; (ii) the maturity of the bond supplied; and, (iii) the sensitivity of the demand by PHI to changes in the real yield.[21]

An increase in the supply of IL bond with certain maturity requires arbitrageurs to absorb the excess supply. Arbitrageurs will add the excess supply into their portfolios if the security issued yields a higher return that compensates for the risk of adding it to the portfolio, that is given by the bond's sensitivity to the real interest rate. This mechanism will push real yields up and thus PHI will also increase their demands of IL bonds. The amount absorbed by PHI will depend on the yield elasticity of their demand. In equilibrium, equation (3.30), the market price of real interest rate risk will be higher after an increase in the supply of IL bond.[22]

Compared to short-term IL bonds, long-term IL bonds' prices are more sensitive to changes in the future real short rate. This implies that arbitrageurs are expose to a higher real short term risk when they have relatively more long-term IL bonds in their portfolios. Thus, the higher the degree of risk aversion and the longer the average maturity of the outstanding IL debt, the higher is the price of real interest rate risk.

Proposition 3.4.7 (Market price of real interest rate risk and the real return of alternative investment opportunities)**.** *The market price of real interest rate risk depends positively on the real return of alternative investment opportunities*

$$\frac{\partial \lambda_{r,t}}{\partial \beta} = \gamma\sigma_r^2 \int_0^T \left[\tau - \frac{\partial A(\tau)}{\partial \beta}\right] \alpha(\tau) B(\tau)\, d\tau \geq 0. \tag{3.33}$$

Proposition 3.4.7 shows that an increase in the real return of alternative investment opportunities positively affects the market price of real interest rate risk and the magnitude of this effect depends on: (i) the degree of risk aversion of arbitrageurs and the real interest rate volatility; and (ii) the sensitivity of the demand by PHI to changes in the

[21] See Appendix A.2.3.

[22] Proposition 3.4.6 rationalises the explanation offered for the puzzling behavior of TIPS yield after the collapse of Lehman Brothers. As commented by Campbell et al. (2009, pp. 105-110) "In 2008, as the subprime crisis intensified, the TIPS yield became highly volatile and appeared to become suddenly disconnected from the yield on nominal Treasuries... Indeed, the sharp peak in the TIPS yield and the accompanying steep drop in the breakeven inflation rate occurred shortly after an event that some observers blame for the anomalous behavior of TIPS yields... The traders at PIMCO saw then a flood of TIPS on the market, for which there appeared to be few buyers. Distressed market makers were not willing to risk taking positions in these TIPS...".

real yield. An increase in the real return of alternative investment opportunities makes PHI decrease their holdings of IL bonds which generates an excess supply of IL bonds. Arbitrageurs will buy the excess supply if the security issued yields a higher real return.

An increase in β can be seen as an increase in the supply of IL bonds since PHI will sell part of their IL bonds holdings after an increase in the real return of alternative investment opportunities. Again, arbitrageurs will add the excess supply into their portfolio if the security issued yields a higher return. In the extreme case, $\alpha \to 0$ which represents the case when there are no PHI, β does not influence the market price of the real interest rate risk.

For the case of the market price of inflation risk we re-express equations (3.11b); with the market-clearing conditions, equations (3.6a)-(3.6b); the demand of PHI, equation (3.5); and bonds' yields and prices, equations (3.1a)-(3.1b) and equations (3.7a)-(3.7b), to obtain

$$\lambda_{\pi,t} = -\gamma\sigma_\pi\rho_{r,\pi}\big[a_0 - a_1\bar{\beta} + a_2\, r_t\big] + \gamma\sigma_\pi Q_\pi, \tag{3.34}$$

where

$$Q_\pi \equiv \sigma_\pi \int_0^T s_\tau^\$ C_\$(\tau)\, d\tau + \rho_{r,\pi}\sigma_r \int_0^T \big[s_\tau B(\tau) + s_\tau^\$ B_\$(\tau)\big]\, d\tau. \tag{3.35a}$$

Equation (3.34) shows that the market price of inflation risk is determined by the total exposure of arbitrageurs' portfolio to inflation risk. The direct exposure to inflation risk is given by arbitrageurs' holding of nominal bonds, while both IL and nominal bonds have a cross-effect to the market price of inflation risk through the correlation between the inflation and the real rate. When $\rho_{r,\pi} = 0$ the cross-effects of market prices of risk are zero.

Equation (3.34) shows that when the correlation between the inflation rate and the instantaneous real interest rate is different from zero, changes in the supply of IL bonds and changes in the real return of alternative investment opportunities will alter the market price of inflation risk by

$$\frac{\partial\lambda_{\pi,t}}{\partial s_\tau} = \gamma\rho_{r,\pi}\sigma_r\sigma_\pi\left[B(\tau) - \int_0^T \frac{\partial A(\tau)}{\partial s_\tau}\alpha(\tau)B(\tau)\, d\tau\right], \tag{3.36a}$$

$$\frac{\partial\lambda_{\pi,t}}{\partial\bar{\beta}} = \gamma\rho_{r,\pi}\sigma_r\sigma_\pi\left[\int_0^T \left[\tau - \frac{\partial A(\tau)}{\partial\bar{\beta}}\right]\alpha(\tau)B(\tau)\, d\tau\right]. \tag{3.36b}$$

When $\rho_{r,\pi} > 0$, by adding the excess supply of IL bonds arbitrageurs are increasing not only real rate risk but also inflation risk in their portfolios. The intuition is that a portfolio with a certain amount of nominal and IL bonds is riskier when the inflation and the instantaneous real interest rate have positive comovements than when the inflation and the instantaneous real interest rate are independent or have negative correlation. Then, in equilibrium the market price of inflation risk is positively (negatively) affected by changes in the excess supply of IL bonds when the correlation between the inflation and the instantaneous real interest rate is positive (negative).[23]

The supply of nominal bonds may affect both the market price of real interest rate risk and the market price of inflation risk,

$$\frac{\partial \lambda_{r,t}}{\partial s^{\$}_{\tau}} = \gamma\sigma_r\Big[\sigma_r B_{\$}(\tau) + \rho_{r,\pi}\sigma_\pi C_{\$}(\tau)\Big] - \gamma\sigma_r^2 \int_0^T \frac{\partial A(\tau)}{\partial s^{\$}_{\tau}}\alpha(\tau)B(\tau)\, d\tau, \tag{3.37a}$$

$$\frac{\partial \lambda_{\pi,t}}{\partial s^{\$}_{\tau}} = \gamma\sigma_\pi\Big[\sigma_\pi C_{\$}(\tau) + \rho_{r,\pi}\sigma_r B_{\$}(\tau)\Big] - \gamma\rho_{r,\pi}\sigma_r\sigma_\pi \int_0^T \frac{\partial A(\tau)}{\partial s^{\$}_{\tau}}\alpha(\tau)B(\tau)\, d\tau. \tag{3.37b}$$

When $\rho_{r,\pi} = 0$, nominal bond supply positively affects the market prices of real interest rate risk and inflation risk. By adding nominal bonds to the portfolio arbitrageurs are increasing their portfolio's exposure to real interest rate risk and inflation risk which results in higher prices of risk in equilibrium. The portfolio's risk increases when $\rho_{r,\pi} > 0$, and then market prices of risk are higher than when $\rho_{r,\pi} = 0$. Finally, when $\rho_{r,\pi} < 0$ the final effect of nominal bond supply on market prices of risk depend on the ratio of the nominal bond return's sensitivity with respect to the real rate and inflation uncertainty.

In this subsection we find that market prices of real interest risk and inflation risk are determined by the total exposure of arbitrageurs' portfolio to each source of risk. This representation allows us, for example, to understand the effect of debt issued by Treasuries or open market operations carried by central banks on market prices of risks. For instance, Proposition 3.4.6 states that the market price of real interest rate risk is positively affected by an increase in the supply of IL bonds. We also see that a change in the supply of IL bonds or in the real return of alternative investment opportunities may affect the inflation risk premium when $\rho_{r,\pi} \neq 0$. Finally, the effect of an increase in the supply of nominal bonds in market prices of risk depends on the correlation between the real interest rate and the inflation.

3.5 Implications

It is argued that IL government bonds provide benefits to fiscal and monetary policymakers, see for example Shen (1995), Barr and Campbell (1997), and Deacon et al. (2004). From the fiscal point of view, by issuing IL bonds instead of nominal ones a government may reduce borrowing costs by not having to pay the inflation risk premium. From monetary policymakers perspective, the BEI is used to extract instantaneous market information about long-term inflation expectations which contributes to enhance the effectiveness

[23]It is argued that by introducing IL bonds, a government can improve market information mechanisms and enhance the credibility of the monetary policy because their issuance incentivizes the government to take an active role in controlling inflation. Thus, the issuance of IL bonds should reduce the market price of inflation risk. The model predicts this fact when the correlation between the inflation and the instantaneous real interest rate is negative, $\rho_{r,\pi} < 0$.

of central banks' decisions.

15 years after the first issuance of TIPS, there is still no consensus among policymakers and academics on what concrete benefits TIPS provide to the government, see Dudley et al. (2009) and Fleckenstein et al. (2010). The presence of time-varying premia (inflation and liquidity risk), and price pressures by institutional factors (demand and supply shocks) pose doubts regarding the theoretical benefits of IL bonds from the government perspective.[24]

For instance, Fleckenstein et al. (2010) present the TIPS-Treasury bond puzzle by showing that nominal US Treasury bonds are consistently overvalued relative to TIPS. Several studies explain that the underpricing of TIPS is due to the presence of a positive liquidity premium in TIPS real yield.[25] In terms of financing expected costs, when the inflation risk premium is higher than the liquidity premium the Treasury will save money by issuing IL instead of nominal bonds of comparable duration. In the opposite case, the Treasury will face ex-ante higher costs by issuing IL when the liquidity premium is higher than the inflation risk premium.

In December 2007 Dudley's speech on the FRBNY accounts for the cost and benefit of the TIPS program and underlines other potential benefits of issuing IL bonds apart from the difference in funding cost such as diversification benefits of the Treasury's funding sources, access to a market-determined measure of inflation expectations, and the provision of a risk-free asset to long-term investors.[26] In this section we analyse the potential benefits of an IL program taking into account the implications of the model presented in this chapter.

3.5.1 Can IL bond issuance save the inflation risk premium?

To start an analysis of the benefits of issuing IL bonds in terms of financing expected costs, we assume that there is no liquidity differential between nominal and IL bonds. That is, investors face only two sources of uncertainty, real rate and inflation risk.

A standard two-factor Vasicek model would predict the BEI equals the expected rate of inflation plus a positive inflation risk premium which increases at longer-time horizons. In the preferred-habitat model, in contrast, when arbitrageurs are risk-averse the inflation risk premium may be negative. Remember that in the preferred-habitat model the market price of risk is determined by the total exposure of arbitrageurs' portfolio to each source of risk. When the main source of risk is given by the real rate risk, and the correlation

[24] Evidence of time-varying risk premia can be found in Sack and Elsasser (2004), Ang et al. (2008), D'Amico et al. (2010), Campbell et al. (2009) and Christensen and Gillan (2011). Greenwood and Vayanos (2010) provide evidence that the maturity structure of government debt affects bonds yields and excess returns.

[25] See for example Sack and Elsasser (2004), Roush (2008), Dudley et al. (2009), D'Amico et al. (2010), Fleming and Krishnan (2009), Viceira and Pflueger (2011), among others.

[26] See http://www.ny.frb.org/newsevents/speeches/2007/dud071213.html

between the inflation rate and the real rate is negative, inflation will hedge the real rate risk of arbitrageurs' portfolio and inflation exposure will present a negative inflation risk premium.

Proposition 3.4.6 states that the supply of IL bonds positively affects the market price of real interest rate risk, and the magnitude of these effects depends positively on the degree of risk aversion of arbitrageurs, the real interest rate volatility, and the average maturity of the outstanding IL debt. Therefore, the common agreement that by issuing IL bonds a government can save the inflation risk premium is not always true. Indeed, when the main source of risk of arbitrageurs' portfolio is given by the real interest rate risk, inflation may behave as a portfolio hedger when its correlation with the real interest rate is negative.

3.5.2 Does the issuance of IL provides diversification benefits to the Treasury?

An increase in the net supply of a particular type of debt by the Treasury may lead to higher financing costs and a high exposure to the risk factor affecting that debt. In terms of diversification benefits the issuance of IL bonds may help to reduce expected borrowing costs of the Treasury and the variability of its financial debt position.

The standard no-arbitrage term structure models used to reproduce the dynamics of real and nominal yields are based on the assumption of a representative-agent model where demand and supply effects do not play any role. In contrast, the preferred-habitat model provides information about how bond demand and supply changes affects real and nominal yields.In the model, debt supply and demand changes affects real and nominal yields through the market prices of risk. An increase in the supply of any type of bond with certain maturity requires arbitrageurs to absorb the excess supply. Arbitrageurs will add the excess supply into their portfolios if the security issued yields a higher return that compensate for its risk. Then, the issuance of IL bonds will help the Treasury for diversification purposes under two different conditions.

First, when the correlation between the real rate and the inflation rate is non-negative, by issuing IL bonds instead nominal bonds of comparable term to mature the Treasury will reduce borrowing costs for two reasons. The presence of PHI will leave less amount of IL bonds to be absorbed by arbitrageurs which results in a lower market price of real rate risk as compared with a nominal bond issuance.[27] The second reason is that by issuing IL bonds instead nominal bonds the Treasury will save the inflation risk premium. When the inflation rate and the real rate are positively correlated the exposure of nominal bond holdings to inflation has a positive cross-effect over the real rate risk premium.

[27] In the extreme case when the demand of PHI is infinitely elastic the market price of real interest rate risk will not be altered after an increase in the supply of IL bonds.

Second, when the main source of risk of arbitrageurs' portfolio is given by inflation, and there is a negative correlation between real interest rate and inflation, it will be recommendable to issue IL bonds instead of comparable nominal bonds. In this situation, the government will obtain diversification benefits by issuing IL bonds not only by reducing expected borrowing costs but also from minimizing the variability of its financial outstanding debt position.

3.5.3 Does the BEI provide a market measure of inflation expectations?

It is argued that the BEI provides instantaneous market information about long-term inflation expectations which help policymakers to enhance monetary policy decisions. However, there is no agreement about how to correctly extract this information from BEI. Our model predicts that when arbitrageurs are risk-neutral, EH holds, bond risk premia are zero, and forward BEI react one-for-one to changes in expected inflation. That is, real forward rates react one-for-one to changes in expected spot rates, and nominal forward rates react one-for-one to changes in expected spot rates and to changes in expected inflation. In other words, arbitrageurs incorporate all information about expected inflation rates into forward BEI.

Vayanos and Vila (2009) show that in the presence of PHI and risk-averse arbitrageurs, forward real rates under-react to changes in expected spot real rates. In this article we find that forward nominal rates might react differently from real forward rates to changes in the real short rate, see Proposition 3.4.4. When inflation and the real short rate are positively correlated, nominal bonds are riskier than IL bonds. Thus, compared to real forward rates, the under-reaction of nominal forward rates to changes in the real short rate will be stronger. On the contrary, if the inflation and the real short rate are negatively correlated, then the under-reaction of nominal forward rates will be weaker. The result is that the forward BEI will include the expected inflation rate, the inflation risk premium, and a term proportional to the real short rate. The last term appears as a consequence of the correlation between the inflation and the real interest rate.

Then, in periods of financial distress when arbitrageurs are more risk-averse, it is expected the long-term BEI includes a risk premium (apart from the inflation risk premium required by buy-and-hold investors) that depends on the correlation between inflation and the real interest rate. Since arbitrageurs care about short term bond returns instead of bond yields, the forward BEI may not adequately capture the compensation that buy-and-hold investors demand to cover the expected rate of inflation.

3.6 Concluding Remarks

In this article we study a theoretical term structure of BEI in a preferred-habitat framework. The model assumes the term structure of real and nominal yields comes from the interaction between buy-and-hold or PHI who demand IL bonds with specific maturity and risk-averse arbitrageurs (mark-to-market investors) who allocate their real wealth in IL and nominal bonds based on an instantaneous mean-variance problem. We extend Vayanos and Vila (2009) by adding inflation into the model and by distinguishing between the real and the nominal term structure of interest rates in order to understand the bond markets pricing mechanism predicted by the framework.

In the absence of PHI, the model is equivalent to the standard two-factor Vasicek (1977) model where market prices of risk are determined by the arbitrageurs' risk aversion coefficient and the total outstanding level of IL and nominal debt relative to the arbitrageurs' level of real wealth. When PHI participate in the IL bond market, bonds' risk premiums are determined by the total exposure of arbitrageurs' portfolio to each source of risk. The main implication of this result is that the inflation risk premium can be negative. For instance, when the main source of risk of arbitrageurs' portfolio is given by the real interest rate, inflation may behave as a portfolio hedger when its correlation with the real interest rate is negative. We also find that forward nominal rates might react differently from real forward ones to changes in the real short rate. This result implies that when arbitrageurs are highly risk averse the forward BEI may not adequately capture the compensation that buy-and-hold investors demand to cover the expected rate of inflation.

Chapter 4

Market Inflation and Inflation-Linked Bonds

4.1 Introduction

A fundamental question that arises when studying the benefits of issuing inflation-linked (IL) government bonds is whether the index to which IL bonds are linked accurately reflects inflation exposure of the investors, which we define as the "market" inflation. For instance, Treasury Inflation Protected Securities (TIPS) are IL bonds issued by the U.S. Treasury in which their principal is indexed to the U.S. non-seasonally adjusted consumer price index for all urban consumers (CPI-U). When the degree of investors' exposure to the different components of the CPI-U differs from the weights used to compute the true index, the market inflation will differ from CPI-U inflation and the IL bond will no longer be the real riskless asset.

The absence of arbitrage opportunities in a frictionless market implies that there exists a pricing kernel or stochastic discount factor (SDF) that assigns prices to securities on the basis of their future claims.[1] While the pricing kernel theory is based on real payoffs one can easily transform a given real SDF to price securities with nominal future payoffs by identifying the dynamics of the proper price index. In this chapter we propose that the price index at which investors deflate nominal payoff may differ from the price index used to adjust IL bonds which in the particular case of TIPS is the CPI-U.

Therefore, we develop and estimate a no-arbitrage term structure model that fits U.S. nominal bonds data in order to obtain a measure of the U.S. expected market inflation as the weighted average of the main component of the CPI-U inflation. Our main goal is to address whether the U.S. expected market inflation differs from the inflation index used for TIPS and determine the main implications for the benefits of issuing IL bonds. Specifically, our research question is oriented in studying bonds risk premiums implied by the model and whether the indexation rule can be improved from a government and/or an investor point of view.

To estimate the market inflation, we introduce an exogenous process for the price level such that the expected market inflation is a weighted (unknown) average of expected core, food, and energy inflation which are assumed to be three correlated Ornstein-Uhlenbeck processes. A recent article by Ajello et al. (2011) estimates a model for nominal and real term structures in which underscores the advantages of modelling the dynamics of the individual inflation components. We differ from them in the general specification of the model and since they assumed that the weights used to compute the market inflation rate are given by a known index.[2]

The main feature that our model handles is that it allows for time-varying risk premiums by assuming a stochastic variation in the real investment opportunities as in the intertemporal capital asset pricing model (ICAPM) of Merton (1973). Nielsen and Vas-

[1] See Ross (1976) and Harrison and Kreps (1979), Hansen and Renault (2009).

[2] Ajello et al. (2011) focus on the term structure models' ability to forecast CPI and the Personal Consumption Expenditure (PCE) inflation.

salou (2002) show that in the ICAPM investors need to hedge only against changes in the random position (real interest rate) and slope (Sharpe ratio) of the instantaneous capital market line (CML). Thus, we introduce a model in which the real investment opportunity set is fully described by the time variation in the real interest rate and the maximum Sharpe ratio of the economy which follow two correlated Ornstein-Uhlenbeck processes, as in Brennan et al. (2004), Brennan and Xia (2006) and Lettau and Wachter (2007, 2011).

We express the model in a state space form and then we run the Kalman filter to estimate the time series of the state variables. We use U.S. monthly data on nominal bonds, inflation components (core, food and energy) and stock market from February 1957 to December 2011. We estimate the time series of the market inflation as the weighted average of the expected core inflation ($\hat{\omega}_c = 0.55$), food inflation ($\hat{\omega}_f = 0.175$), and energy inflation ($\hat{\omega}_e = 0.275$). All weights significantly differ from the weights of the CPI-U (known as headline) inflation $(\omega_c^h; \omega_f^h; \omega_e^h) = (0.675; 0.235; 0.09)$ except the energy component in which we can not reject the null hypothesis because of the high standard error of its parameter estimate. We find that expected energy inflation is the most volatile component of market inflation but it is uncorrelated with the real SDF implying that there does not appear to be a risk premium associated with the energy inflation component. In contrast, expected core and food inflation are significantly correlated with the SDF implying that the inflation risk premium is the weighted sum of core and food inflation risk premiums.

We find time-varying risk premiums on real zero-coupon bonds over the real riskless rate and they are (negatively) proportional to the maximum Sharpe ratio of the economy. This suggests that when the Sharpe ratio is increasing, premiums on real bonds are decreasing denoting that these assets serve as hedgers to changes in the investment opportunity set. For the case of nominal bonds, we get that the average risk premiums over the real riskless rate present a hump-shaped curve with negative values for short- and medium-term maturities and positive risk premium for long-term nominal bonds. This implies that on average only long-term nominal bonds carry a positive inflation risk premium over the real risk-free rate. This premium is largely explained by the food component reaching more than 35 basis points (*bp*) for nominal bonds with maturities longer than 10 years. Finally, we find that the risk premiums on nominal bonds over real bonds with the same maturity are time-varying and their signs change through time. For instance, during the beginning of the last financial crisis the difference in the risk premiums was around $-100bp$. That is, the Treasury by issuing 10-year real bonds in that period instead of nominal bonds with the same maturity have increased their financial cost in 1% per every dollar of the new real debt issued.

IL government bonds are argued to provide benefits to different players in the economy. From the investors' point of view, IL bonds are the riskless assets in real terms for buy-and-hold long-term investors whose investment horizon perfectly matches the maturity of the IL bond, see Campbell and Viceira (2001), Brennan and Xia (2002), Campbell et al. (2003) and Wachter (2003). However, in the case of TIPS we find an inflation basis risk included in TIPS indexation rule due to the fact that market inflation differs from the inflation index to which TIPS are linked. From the fiscal point of view, by issuing IL bonds instead of nominal ones a government may reduce borrowing costs by not having to pay the inflation risk premium, see for example Shen (1995), Barr and Campbell (1997), and Deacon et al. (2004). We find that this argument is not always true since the difference in premium between nominal and real bonds with the same maturity is not only time-varying but also the sign changes through time.

Our model is constructed following different aspects related with term structure models of interest rates. First, Duffee (2010) shows that conditional maximum Sharpe ratios implied by fully flexible Gaussian term structure models of interest rate are not realistic since they are astronomically high. To solve the issue he constrains the maximum Sharpe ratio to be lower than an upper bound. In our model, we identify the maximum Sharpe ratio on the economy with the Sharpe ratio on the stock market including the latter as an observed variable in the measurement equation of the state space model. We are implicitly assuming that shocks to the market portfolio are perfectly correlated with the real SDF. Brennan et al. (2004) constrain the long-run mean of the maximum Sharpe ratio to 0.7 (the observed value reported for the stock market is 0.5) and find that the estimated Sharpe ratio is significantly correlated with the ex-post equity market Sharpe ratio. Our estimate gives a mean of the estimated maximum annual Sharpe ratio of 0.28 with a maximum value of 2.97 which are realistic values considering previous studies, see Duffee (2010).

Second, recent research on the construction and estimation on term structure models documents that there exist factors that contain information about investors' expectations of future yields which is absent in the cross-section of current yields, see Cochrane and Piazzesi (2005), Cooper and Priestley (2009), Ludvigson and Ng (2009), Joslin et al. (2010), Duffee (2011), Ajello et al. (2011) and Chernov and Mueller (2012). Duffee (2011) calls these variables "hidden" factors and he defines them as those factors that play an important role in determining the variation of bonds' risk premiums yet have no effect on current yields. He concludes that the importance of hidden factors on risk premiums is strong both statistically and economically. By disentangling inflation components as in Ajello et al. (2011), we are identifying factors not included in yield data which help to improve the estimates of bonds' risk premiums.

Finally, as it was pointed by Lintner (1975), bonds' risk premium, particularly the inflation risk premium, cannot be analyzed by solely considering nominal and IL bonds; all assets in the economy, specially those that are potential hedgers against inflation, will affect the inflation risk premium.[3] Thus, we simultaneously estimate the model using bond

[3] "... the certainty-equivalent real return on any asset with fixed money return will differ form the excess of its nominal return over the expected rate of purchasing power loss (the Fisher rule) by amounts which increase with risk aversion and also with asset's marginal portfolio risks in real terms (the entire "row-sum" of its variance and its covariances of purchasing power risks with all other assets in the portfolio)." John Lintner (1975).

and stock market data, see for instance Campbell et al. (2009), Campbell et al. (2009), and Lettau and Wachter (2011).[4] In Campbell et al. (2009) and Campbell et al. (2009), they allow for a time-varying: i) real risk-free rate, iii) volatility of asset's return, and iii) correlation between the asset and the real SDF; leaving the maximum Sharpe ratio of the economy to be constant. This implies that the investor's investment opportunity set is totally described by the random position but not the slope of CML. Since the slope of the CML also changes investors need to hedge not only changes in the the real riskless rate but also the maximum Sharpe ratio.[5] Lettau and Wachter (2011) allow for a time-varying real interest rate and maximum Sharpe ratio, as we do, but they implicitly assume that the market inflation is equal to CPI-U inflation.

The structure of the article is as follows. Section 4.2 presents the dynamics of the investment opportunity set and derives the equilibrium term structure of real and nominal interest rates that results from the model. Section 4.3 presents the data set and the technique used to estimate our model. Section 4.4 discusses the empirical results. Section 4.5 concludes.

4.2 The Model

In this section we introduce a model in which the real investment opportunity set is fully described by the time variation in the real interest rate and the maximum Sharpe ratio of the economy. The absence of arbitrage opportunities in a frictionless market implies the existence of a real pricing kernel or stochastic discount factor (SDF), M_t, such that $\mathrm{E}_t\big[d(M_t P_{j,t})\big] = 0$, where $P_{j,t}$ is the price on any non-dividend paying asset. The previous condition implies the asset's instantaneous expected real return can be written as

$$\begin{aligned}\mathrm{E}_t\Big[\frac{dP_{j,t}}{P_{j,t}}\Big] &= -\mathrm{E}_t\Big[\frac{dM_t}{M_t}\Big] - \mathrm{Cov}_t\Big[\frac{dM_t}{M_t}, \frac{dP_{j,t}}{P_{j,t}}\Big] \\ &= -[\mu_m + \sigma_m\, \sigma_j\, \rho_{m,j}]\, dt, \end{aligned} \tag{4.1}$$

where μ_m and σ_m represent the drift and volatility of the real SDF, σ_j is the volatility of the real return of asset j, and $\rho_{m,j}$ is its correlation with the real SDF. It can be easily shown that $-\mu_m$ is the real risk-free rate and $-\sigma_m$ is the maximum Sharpe ratio of the economy. Then, the dynamics of assets' expected returns, or equivalently assets' prices, will be governed by four different factors: i) the real risk-free rate, ii) the maximum Sharpe ratio, iii) the volatility of asset's return, and iv) the correlation between the asset and the real SDF.

[4]Previous articles related with our study using only market bond data are D'Amico et al. (2010), Christensen et al. (2010), Christensen and Gillan (2011), Haubrich et al. (2011), among others.

[5]See for instance Tang and Whitelaw (2011) which document predictable time-variation in stock market Sharpe ratios.

4.2.1 Real risk-free rate, the maximum Sharpe ratio and the real SDF

Let r_t be the instantaneous real risk-free rate which is assumed to follow an Ornstein-Uhlenbeck process

$$dr_t = \kappa_r[\bar{r} - r_t]dt + \sigma_r dZ_{r,t}, \tag{4.2}$$

where κ_r is the mean-reverting parameter towards the unconditional long-run mean $\bar{r}$; the parameter σ_r represents the volatility or diffusion coefficient; and $Z_{r,t}$ is a standard Brownian motion representing particular shocks to the instantaneous real risk-free rate.

The maximum Sharpe ratio of the economy is η_t and it is also assumed to follow an Ornstein-Uhlenbeck process

$$d\eta_t = \kappa_\eta[\bar{\eta} - \eta_t]dt + \sigma_\eta dZ_{\eta,t}, \tag{4.3}$$

where κ_η is the mean-reverting parameter towards the unconditional long-run mean $\bar{\eta}$; the parameter σ_η is the volatility coefficient; and $Z_{\eta,t}$ is a standard Brownian motion representing particular shocks to the maximum Sharpe ratio of the economy which might be correlated with shocks to the instantaneous real risk-free rate $dZ_{\eta,t}dZ_{r,t} = \rho_{\eta r}dt$. The variable η_t plays a key point in the model since it determines the time-varying price of risk of the economy.

The real SDF (M_t) which determines the asset pricing properties of each security is given by

$$\frac{dM_t}{M_t} = -r_t dt - \eta_t dZ_{m,t}. \tag{4.4}$$

Here $Z_{m,t}$ is a standard Brownian motion referring to particular shocks to the fundamentals of the economy which might be correlated either with shocks to the instantaneous real risk-free rate $dZ_{m,t}dZ_{r,t} = \rho_{mr}dt$ or with shocks to the maximum Sharpe $dZ_{m,t}dZ_{\eta,t} = \rho_{m\eta}dt$. Finally, assets' expected returns can be express as

$$\mathrm{E}_t\left[\frac{dP_{j,t}}{P_{j,t}}\right] = r_t dt + \eta_t\sigma_j\rho_{m,j}dt, \tag{4.5}$$

where $\eta_t\sigma_j\rho_{m,j}$ represents asset j's risk premium over the real risk-free rate.

4.2.2 Inflation and the nominal SDF

Because we are interested in discovering the market inflation and the inflation risk premium we need to specify the process for inflation. Let Q_t denote the process for the price level index in which its dynamic is assumed to be

$$\frac{dQ_t}{Q_t} = \pi_t dt + \sigma_q dZ_{q,t}, \tag{4.6}$$

where π_t refers to the expected rate of market inflation; σ_q is the volatility of the price level index; and, $Z_{q,t}$ refers to a standard Brownian motion representing particular shocks to the price level index which might be correlated to other shocks in the economy, $dZ_{i,t}dZ_{q,t} = \rho_{iq}dt$.

The question we address in this chapter is whether the price level index of the economy accurately reflects the cumulative inflation given by the CPI-U, the index to which TIPS are indexed. Investors may differ in the degree to which they are exposed to different components of the CPI-U index. For this reason, we express the expected rate of market inflation (π_t) as a weighted average of core inflation ($\pi_{c,t}$), food inflation ($\pi_{f,t}$), and energy inflation ($\pi_{e,t}$)

$$\pi_t = \omega_c \pi_{c,t} + \omega_e \pi_{e,t} + \omega_f \pi_{f,t}, \tag{4.7}$$

where ω_i represents the weight of the ith component in the expected market inflation. The components are assumed to follow three correlated Ornstein-Uhlenbeck processes given by

$$d\pi_{c,t} = \kappa_{\pi_c}[\bar{\pi}_c - \pi_{c,t}]dt + \sigma_{\pi_c} dZ_{\pi_c,t}, \tag{4.8a}$$
$$d\pi_{f,t} = \kappa_{\pi_f}[\bar{\pi}_f - \pi_{f,t}]dt + \sigma_{\pi_f} dZ_{\pi_f,t}, \tag{4.8b}$$
$$d\pi_{e,t} = \kappa_{\pi_e}[\bar{\pi}_e - \pi_{e,t}]dt + \sigma_{\pi_e} dZ_{\pi_e,t}. \tag{4.8c}$$

Here $(\kappa_{\pi_c}, \kappa_{\pi_f}, \kappa_{\pi_e})$ are the mean-reverting parameters toward the unconditional long-run means $(\bar{\pi}_c, \bar{\pi}_f, \bar{\pi}_e)$; the parameters $(\sigma_{\pi_c}, \sigma_{\pi_f}, \sigma_{\pi_e})$ represent the volatility coefficients; and $(Z_{\pi_c,t}, Z_{\pi_f,t}, Z_{\pi_e,t})$ are three correlated standard Brownian motions with $dZ_{\pi_i,t}dZ_{\pi_j,t} = \rho_{\pi_i\pi_j}dt$.

The nominal SDF ($M_t^\$$) is related to the real SDF and the price level index by $M_t^\$ = M_t Q_t^{-1}$ which results that the dynamics of $M_t^\$$ are given by[6]

$$\frac{dM_t^\$}{M_t^\$} = -i_t dt - \eta_t dZ_{m,t} - \sigma_q dZ_{q,t}, \tag{4.9}$$

where

$$i_t = r_t + \pi_t - \sigma_q^2 - \eta_t \sigma_q \rho_{mq}, \tag{4.10}$$

is the instantaneous nominal rate. That is, the instantaneous nominal short rate is equal to the real instantaneous risk-free rate plus the expected rate of inflation adjusted by the

[6]See Appendix B.1.1.

instantaneous inflation risk premium which is given the correlation between the price level index and the real SDF.

4.2.3 Prices and returns on Treasury bonds

Real bonds

We define real bonds to hypothetical zero-coupon IL bonds which are indexed to the market inflation without any inflation indexation lag. Let $P_{t,\tau}$ be the price at time t of a real zero-coupon bond with a defined term to be redeemed $\tau \in (0, T]$. At maturity, a real bond pays 1 unit of real wealth. Thus, these bonds represent the riskless asset in real terms for buy-and-hold long-term investors whose investment horizon perfectly matches the maturity of real bonds.[7]

The absence of arbitrage opportunities implies $\mathrm{E}_t[d(M_t P_{t,\tau})] = 0$, or

$$\mathrm{E}_t\left[\frac{d(M_t P_{t,\tau})}{M_t P_{t,\tau}}\right] = \mathrm{E}_t\left[\frac{dP_{t,\tau}}{P_{t,\tau}}\right] + \mathrm{E}_t\left[\frac{dM_t}{M_t}\right] + \mathrm{E}_t\left[\frac{dM_t}{M_t}\frac{dP_{t,\tau}}{P_{t,\tau}}\right] = 0. \tag{4.11}$$

Appendix B.1.2 shows that the price of real bonds can be expressed as

$$P_{t,\tau} = \exp\Big(- \big[A(\tau) + B(\tau)\eta_t + C(\tau) r_t\big]\Big), \tag{4.12}$$

with the boundary condition $A(0) = B(0) = C(0) = 0$. The coefficients on the maximum Sharpe ratio and the real risk-free rate are given by

$$B(\tau) = -\frac{\sigma_r \rho_{mr}}{\kappa_r \kappa_\eta^*} + \frac{\sigma_r \rho_{mr}}{\kappa_r \kappa_\eta^*} e^{-\kappa_\eta^* \tau} + \frac{\sigma_r \rho_{mr}}{(\kappa_\eta^* - \kappa_r)\kappa_r}\big[e^{-\kappa_r \tau} - e^{-\kappa_\eta^* \tau}\big], \tag{4.13}$$

$$C(\tau) = \frac{1 - e^{-\kappa_r \tau}}{\kappa_r}, \tag{4.14}$$

with $\kappa_\eta^* = \kappa_\eta + \rho_{m\eta}\sigma_\eta$, whereas the constant term $A(\tau)$ is defined in (B.12). The yield to maturity on a real bond is defined as

$$y_{t,\tau} = -\frac{\log(P_{t,\tau})}{\tau} = \frac{1}{\tau}\Big[A(\tau) + B(\tau)\eta_t + C(\tau) r_t\Big], \tag{4.15}$$

which is an affine function of on the maximum Sharpe ratio and the real risk-free rate. When the real interest rate presents some degree of persistence ($\kappa_r > 0$), an increase in the real risk-free rate positively affects the yield of IL bonds and the magnitude of the effect is decreasing in the maturity of the bond. If real yields include positive risk premiums, then $B(\tau) > 0$ which implies that an increase in the reward per unit of risk in the economy (η) pushes real yields up. On the contrary, if real yields contain negative

[7]See Brennan and Xia (2002), Wachter (2003), and Cartea et al. (2012).

risk premiums over the instantaneous riskless rate, an increase in η will reduce real yields. When factors affecting real yields are uncorrelated with the real SDF ($\rho_{m\eta} = \rho_{mr} = 0$), real bonds will not bear any risk and real yields will not be affected by η_t.

The instantaneous expected return on a real bond with maturity τ over the instantaneous real risk-free rate is given by

$$\mathrm{E}_t\left[\frac{dP_{t,\tau}}{P_{t,\tau}} - r_t\right] = \Big[B(\tau)\sigma_\eta\rho_{m\eta} + C(\tau)\sigma_r\rho_{mr}\Big]\eta_t. \tag{4.16}$$

which represents the risk premiums on real bonds. Equation (4.16) shows that changes in the maximum Sharpe ratio creates a time-varying behaviour of real bonds' risk premiums. As it is stated in equation (4.5), the level of the real bond's risk premium is determined by the covariance between the return of the real bond and the real SDF.

Nominal bonds

Let $P^{\$}_{t,\tau}$ be the price at time t of a nominal zero-coupon bond with a defined term to be redeemed $\tau \in (0, T]$. At maturity, a nominal bonds pays 1 unit of currency. If a nominal bond is carried to expiration it will yield a nominal riskless return, free of real interest rate risk but exposed to inflation risk.

The absence of arbitrage opportunities implies $\mathrm{E}_t[d(M^{\$}_t P^{\$}_{t,\tau})] = 0$, or

$$\mathrm{E}_t\left[\frac{d(M^{\$}_t P^{\$}_{t,\tau})}{M^{\$}_t P^{\$}_{t,\tau}}\right] = \mathrm{E}_t\left[\frac{dP^{\$}_{t,\tau}}{P^{\$}_{t,\tau}}\right] + \mathrm{E}_t\left[\frac{dM^{\$}_t}{M^{\$}_t}\right] + \mathrm{E}_t\left[\frac{dM^{\$}_t}{M^{\$}_t}\frac{dP^{\$}_{t,\tau}}{P^{\$}_{t,\tau}}\right] = 0. \tag{4.17}$$

Appendix B.1.3 shows that the price of nominal bonds is given by

$$P^{\$}_{t,\tau} = \exp\Big(- \big[A_n(\tau) + B_n(\tau)\eta_t + C_n(\tau)r_t + \sum_{j=c,e,f} D_{n,j}(\tau)\omega_j\pi_{j,t}\big]\Big), \tag{4.18}$$

with the boundary condition $A_n(0) = B_n(0) = C_n(0) = 0$ and $D_{n,j}(0) = 0$ for $j = c, f, e$. The constant term in (4.18) is defined in Appendix B.1.3 while the coefficients on the state variables are

$$B_n(\tau) = b_1 + b_2 e^{-\kappa^*_\eta\tau} + b_3 e^{-\kappa_r\tau} + \sum_{j=c,e,f} b_{4,j} e^{-\kappa_{\pi_j}\tau}, \tag{4.19}$$

$$C_n(\tau) = \frac{1 - e^{-\kappa_r\tau}}{\kappa_r}, \tag{4.20}$$

$$D_{n,j}(\tau) = \frac{1 - e^{-\kappa_{\pi_j}\tau}}{\kappa_{\pi_j}} \qquad \text{for } j = c, e, f, \tag{4.21}$$

where

$$\kappa_\eta^* = \kappa_\eta + \rho_{m\eta}\sigma_\eta, \tag{4.22}$$

$$b_1 = -\frac{\sigma_q\rho_{mq}}{\kappa_\eta^*} - \frac{\sigma_r\rho_{mr}}{\kappa_\eta^*\kappa_r} - \sum_{j=c,e,f}\frac{\omega_j\sigma_{\pi_j}\rho_{m\pi_j}}{\kappa_\eta^*\kappa_{\pi_j}}, \tag{4.23}$$

$$b_2 = -b_1 - b_3 - \sum_{j=c,f,e} b_{4,j}, \tag{4.24}$$

$$b_3 = \frac{\sigma_r\rho_{mr}}{(\kappa_{\eta^*} - \kappa_r)\kappa_r}, \tag{4.25}$$

$$b_{4,j} = \frac{\omega_j\sigma_{\pi_j}\rho_{m\pi_j}}{(\kappa_{\eta^*} - \kappa_{\pi_j})\kappa_{\pi_j}} \qquad \text{for } j = c, f, e. \tag{4.26}$$

Equation (4.18) shows that the nominal bond prices are driven by five state factors: the maximum Sharpe ratio, the real risk-free rate, and the three market inflation's components. An increase in the real risk-free rate or inflation's components reduce prices of nominal bonds, and the magnitude of the effect is proportional to the maturity of the bond. The yield to maturity on a nominal bond, denoted by $y^\$_{t,\tau}$, is equal to

$$y^\$_{t,\tau} = -\frac{\log(P^\$_{t,\tau})}{\tau} = \frac{1}{\tau}\Big[A_n(\tau) + B_n(\tau)\eta_t + C_n(\tau)r_t + \sum_{j=c,e,f} D_{n,j}(\tau)\omega_j\pi_{j,t}\Big], \tag{4.27}$$

which is an affine function of the maximum Sharpe ratio, the real risk-free rate and the expected market inflation rate components. When nominal yields include positive risk premiums, then $B_n(\tau) > 0$ which implies that an increase in η will push nominal yields up while the contrary will happen if nominal yields include negative risk premiums.

The instantaneous expected return on a nominal bond with maturity τ over the instantaneous real risk-free rate is given by

$$\begin{aligned}\mathrm{E}_t\Big[\frac{dP^\$_{t,\tau}}{P^\$_{t,\tau}} - r_t - \pi_t\Big] &= \Big[B_n(\tau)\sigma_{q\eta} + C_n(\tau)\sigma_{qr} + \sum_{j=c,e,f} D_{n,j}(\tau)\omega_j\sigma_{q\pi_j}\Big] \\ &\quad + \Big[B_n(\tau)\sigma_\eta\rho_{m\eta} + C_n(\tau)\sigma_r\rho_{mr} + \sum_{j=c,e,f} D_{n,j}(\tau)\omega_j\sigma_{\pi_j}\rho_{m\pi_j} - \sigma_q\rho_{mq}\Big]\eta_t.\end{aligned} \tag{4.28}$$

Here we use the notation σ_{ij} to denote the covariance between shocks affecting factor i and shocks to factor j. Equation (4.28) shows that nominal bonds' expected real excess return over the real risk-free rate is determined by the nominal bonds' risk premiums which is composed by constant and time-varying terms. Constant terms are given by the covariance between unexpected inflation shocks and factors affecting nominal yields. Time-varying terms are determined by the covariance between shocks to the real SDF and factors affecting nominal yields

An important point from the Treasury point of view is the risk premiums on nominal bonds relative to real bonds with the same maturity. From equations (4.16) and (4.28) we get

$$\mathrm{E}_t\left[\frac{dP^{\$}_{t,\tau}}{P^{\$}_{t,\tau}} - \pi_t - \frac{dP_{t,\tau}}{P_{t,\tau}}\right] = \Big[B_n(\tau)\sigma_{q\eta} + C_n(\tau)\sigma_{qr} + \sum_{j=c,e,f} D_{n,j}(\tau)\omega_j\sigma_{q\pi_j}\Big] + \Big[G(\tau)\sigma_\eta\rho_{m\eta} + \sum_{j=c,e,f} D_{n,j}(\tau)\omega_j\sigma_{\pi_j}\rho_{m\pi_j} - \sigma_q\rho_{mq}\Big]\eta_t, \tag{4.29}$$

where $G(\tau) = \big[B_n(\tau) - B(\tau)\big]$. Positive values in equation (4.29) imply that by issuing nominal bonds instead of IL bonds the Treasury will have to pay more financing services (ex-ante) due to the inflation risk premium. On the contrary, negative values will suggest that the inflation exposure of nominal bonds are good instruments for investors to hedge against adverse changes in the fundamentals of the economy. In this last case, nominal bonds will be a cheaper source of financing than those of IL bonds with the same maturity.

4.3 Data and Estimation Technique

4.3.1 Data

The dataset used for estimating the model consist of: nominal yields data, inflation data and stock market data. We collect monthly observations from February 1957 to December 2011. We use nominal yields on constant maturity zero-coupon U.S. Treasury bonds with maturities of 3 and 6 months, and 1, 2, 3, 5, 7, and 10 years. We construct our nominal yields data set by combining McCulloch-Kwon zero-coupon yields dataset from February 1957 to December 1991 and data of Treasury yields constant maturity provided by the U.S. Federal Reserve from January 1992 to December 2011.[8] All rates are given as percentages per annum, and are on a continuous-compounding basis. Panel A.i. in Table 4.1 reports summary statistics for nominal bonds yields data. The panel exhibits the well-known properties of the nominal term structure such as: i) a positive relation between a bond's maturity and its average yield, the positive slope of the yield curve; ii) an inverse relation between a bond's maturity and its yield variance; and iii) a high persistence of yields for all maturities and a positive relation between a bond's maturity and its yield persistence.

Panel B.i. in Table 4.1 reports summary statistics for the inflation data which is measured by the percentage change in the CPI-U that includes: a) all items (headline inflation); b) all items less food and energy (core inflation); c) food items (food inflation); and, d) energy items (energy inflation). All series are seasonally adjusted and are available

[8]McCulloch-Kwon data are downloaded from http://www.econ.ohio-state.edu/jhm/ts/mcckwon/mccull.htm, see McCulloch (1975) and Kwon (1992) for further details. The U.S. Fed data are downloaded from http://www.federalreserve.gov/releases/h15/data.htm#fn11. For information on how the Treasury's yield curve is derived see http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/yieldmethod.aspx.

on the web site of the Bureau of Labor Statistics (BLS).[9] Headline and core inflation show similar statistics patterns regarding sample mean (3.83%; 3.78%), standard deviation (3.78%; 3.08%) and the degree of persistence, $AC(1) = (0.63; 0.64)$. Energy inflation presents the highest sample mean (4.43%) with a volatility (24.36%) six times greater than the volatility of headline inflation and with a lower persistence, $AC(1) = 0.42$. Food inflation is the least persistent component of inflation with a first-order auto-correlation of $AC(1) = 0.31$.

Panel B.ii. in Table 4.1 shows, as expected, that all inflation components (core, food and energy) are highly correlated with the headline inflation, $\rho_{\pi_h,\pi_j} = (0.68; 0.55; 0.67)$. However, the correlation among them is much weaker, specifically the correlation between energy inflation with the rest of the components. Panel B.iii. in the same Table reports the estimation of a constrained regression in order to compute the headline inflation as a weighted average of its main inflation components. The weights we find for Core (0.68), food (0.23) and energy (0.09) are similar to those computed for the CPI-weighted index in Ajello et al. (2011).[10] Although weights' component are not constant through time, the fitted headline inflation measured as a weighted average of the various components is almost identical to the inflation computed from the CPI-U index (headline inflation).

For market data we use excess returns for the value-weighted market index (from CRSP) minus one-month Treasury bill rate (from Ibbotson Associates).[11] Panel C.i. in Table 4.1 shows summary statistics for market excess return, market volatility and the realized Sharpe ratio. The annualized mean excess return over the sample period is 5.87% with an annualized unconditional standard deviation of 15.33%, leading to an annualized Sharpe ratio around 0.38 which is slightly lower that the one found by Brennan et al. (2004) for the period of 1952 to 2002.

With the excess return data we fit an exponential general autoregressive conditional heteroskedastic (EGARCH)[12] model by Nelson (1991) to identify and constraint in equation (4.4) the maximum Sharpe ratio of the economy, as suggested in Duffee (2010). Panel C.ii. in Table 4.1 exhibits the estimation of the coefficients and their standard errors. The conditional volatility presents high persistence and it responds asymmetrically to positive and negative market excess return. Particularly, the asymmetric relation between excess return and volatility is negative ($\gamma = -0.17$) which implies that the volatility, on average, tends to rise (fall) when unexpected excess returns are negative (positive). The realized Sharpe ratio shows a low persistence ($AC(1) = 0.07$) with an unconditional volatility of 3.46%.

[9] http://www.bls.gov/cpi/.

[10] Ajello et al. (2011) construct CPI-weighted index with the same inflation components using quarterly data from 1962Q1 to 2009Q4.

[11] Available from Professor Kenneth French's web site.http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_li

[12] By using the EGARCH, we count not only for time-varying conditional volatility, but also for the leverage effects, empirically found in many works; see Black (1976), Christie (1982), Engle and Ng (1993), Huang (2009), and Aboura and Wagner (2012).

Table 4.1: Summary statistics

We collect monthly observations from February 1957 to December 2011. Panel A reports summary statistics for the U.S. Treasury nominal bond yield data. Panel B exhibits summary statistics for the inflation data which is measured by the percentage change in the CPI-U that includes: a) all items (headline inflation); b) all items excluding food and energy (core inflation); c) food items (food inflation); and, d) energy items (energy inflation). Panel C shows summary statistics for market excess returns, market volatility, and the realized Sharpe ratio.

Panel A: Nominal yields

i. Summary statistics (annualized %)

Bond Maturity	3m	6m	1y	2y	3y	5y	7y	10y
Mean	5.0987	5.3297	5.5117	5.7649	5.9249	6.0842	6.3455	6.4641
Std. dev.	2.9987	3.0332	3.0267	2.9498	2.8731	2.7365	2.6630	2.5792
AC(1)	0.9861	0.9866	0.9875	0.9891	0.9902	0.9891	0.9927	0.9934

Panel B: Inflation

i. Summary statistics (annualized %)

	Headline	Core	Food	Energy
Mean	3.8326	3.7800	3.8229	4.4309
Std. dev.	3.7781	3.0792	5.8380	24.3553
AC(1)	0.6259	0.6421	0.3091	0.4175

ii. Correlations

	Headline	Core	Food	Energy
Headline	1.0000	0.6760	0.5467	0.6650
Core	0.6760	1.0000	0.2714	0.1268
Food	0.5467	0.2714	1.0000	0.1027
Energy	0.6650	0.1268	0.1027	1.0000

iii. Constrained regression on headline inflation $\left(\sum_i \omega_i = 1\right)$

	Core	Food	Energy	R^2
Weights	0.6774	0.2352	0.0874	0.9090
Std. Error	0.0078	0.0077	0.0018	-

Panel C: Stock market

i. Summary statistics (annualized %)

	Excess Ret.	Volatility	Sharpe Ratio
Mean	5.8720	14.9147	0.3942
Std. dev.	15.3257	4.1623	3.4607
AC(1)	0.0859	0.8676	0.0677

ii. EGARCH model estimation $\left(\log \sigma_t^2 = \omega + \beta \log \sigma_{t-1}^2 + \alpha\left[|\varepsilon_{t-1}| - \mathbf{E}(|\varepsilon_{t-1}|)\right] + \gamma \varepsilon_{t-1}\right)$

Variable	ω	β	α	γ
Coeffcient	0.3172	0.1982	0.8897	-0.1669
Std. Error	0.0971	0.0694	0.0347	0.0369

4.3.2 Estimation technique

The estimation procedure is carried out by means of casting the model into an state space framework exploiting the theoretical affine relationship between bond yields and the state variables, $(r_t, \eta_t, \pi_{j,t})$.[13] We estimate the set of parameters by running the Kalman filter which is the most popular technique in the affine term-structure literature and is known for being very useful in situations where the underlying state variables are unobservable as in our case.[14]

We begin with an unobserved system of equations called the transition equation which describes the dynamics of the state variables as they were formulated in the model. The Kalman filter uses the state-space representation to recursively make inferences about the unobserved state variables by conditioning on the observed variables. These equations represent the affine relationship between nominal yields and the state variables, equation (4.27); and the rest of observed variables used to identify the variables in the model. We use the recursive inferences to construct and maximize a log-likelihood function to find the optimal set of parameters.

Our model is stated in continuous time in which the dynamics of the state variables are describes by a set of stochastic differential equations. In order to state the transition equation, we need to discretize the dynamics of the state variables given in equations (4.2), (4.3), (4.7) and (4.8a)-(4.8b) which results in

$$\begin{bmatrix} r_t \\ \eta_t \\ \pi_{c,t} \\ \pi_{f,t} \\ \pi_{e,t} \\ \pi_t \end{bmatrix} = \begin{bmatrix} \bar{r}\left(1-e^{-\kappa_r \Delta t}\right) \\ \bar{\eta}\left(1-e^{-\kappa_\eta \Delta t}\right) \\ \bar{\pi}_c\left(1-e^{-\kappa_{\pi_c} \Delta t}\right) \\ \bar{\pi}_f\left(1-e^{-\kappa_{\pi_f} \Delta t}\right) \\ \bar{\pi}_e\left(1-e^{-\kappa_{\pi_e} \Delta t}\right) \\ 0 \end{bmatrix} + F \begin{bmatrix} r_{t-1} \\ \eta_{t-1} \\ \pi_{c,t-1} \\ \pi_{f,t-1} \\ \pi_{e,t-1} \\ \pi_{t-1} \end{bmatrix} + R \begin{bmatrix} \epsilon_{r,t} \\ \epsilon_{\eta,t} \\ \epsilon_{\pi_c,t} \\ \epsilon_{\pi_f,t} \\ \epsilon_{\pi_e,t} \end{bmatrix}, \tag{4.30}$$

where

$$F = \begin{bmatrix} e^{-\kappa_r \Delta t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-\kappa_\eta \Delta t} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-\kappa_{\pi_c} \Delta t} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-\kappa_{\pi_f} \Delta t} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\kappa_{\pi_e} \Delta t} & 0 \\ 0 & 0 & w_c & w_f & w_e & 0 \end{bmatrix}; \quad R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The error terms are assumed $\epsilon_{j,t}|\mathcal{F}_{t-1} \sim \mathcal{N}(0, Q)$ in which the elements of the covariance matrix are given by $Q_{ij} = \frac{\sigma_i \sigma_j \rho_{ij}}{\kappa_i + \kappa_j}\left[1 - e^{-(\kappa_i+\kappa_j)\Delta t}\right]$ for $i, j = r, \eta, \pi_c, \pi_f, \pi_e$.

In order to construct our measurement equation, we use the nominal zero-coupon yields data and their relationship with the state variables stated in equation (4.17) plus

[13] See Harvey (1989).
[14] See Lund (1997) and Bolder (2001).

a measurement error

$$y^{\$}_{t,\tau} = \frac{1}{\tau}\Big[A_n(\tau) + B_n(\tau)\eta_t + C_n(\tau)r_t + \sum_{j=c,e,f} D_{n,j}(\tau)\omega_j\pi_{j,t}\Big] + \varepsilon_{\tau,t}. \tag{4.31}$$

We also identify the realized Sharpe ratio and the realized inflation components

$$\eta^r_t = \eta_t + \varepsilon_{\eta^r,t}, \tag{4.32a}$$
$$\pi^r_{c,t} = \pi_{c,t} + \varepsilon_{\pi^r_c,t}, \tag{4.32b}$$
$$\pi^r_{f,t} = \pi_{f,t} + \varepsilon_{\pi^r_f,t}, \tag{4.32c}$$
$$\pi^r_{e,t} = \pi_{e,t} + \varepsilon_{\pi^r_e,t}. \tag{4.32d}$$

The measurement error terms are assumed to be normally distributed with zero mean and constant covariances matrix as given by

$$H = \begin{bmatrix} \sigma^2/\tau_1 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \sigma^2/\tau_2 & \cdots & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & & & \vdots \\ 0 & 0 & \cdots & \sigma^2/\tau_n & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & & \vdots & \sigma^2_{\eta^r} & & & \vdots \\ \vdots & \vdots & & \vdots & & \sigma^2_{\pi_c} & & \vdots \\ \vdots & \vdots & & \vdots & & & \sigma^2_{\pi_f} & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \cdots & \cdots & \sigma^2_{\pi_e} \end{bmatrix}.$$

4.4 Empirical Results

4.4.1 State variables

In this section we present the estimates of the state space model presented in Section 4.3.2. In the transition equations we fix unconditional means of all state variables but the real interest rate to match their unconditional sample moments. In particular, we assume that the long-run mean of the maximum Sharpe ratio is equal to the sample mean of the realized Sharpe ratio on the market $\bar{\eta} = 0.39$; see Panel C.i. in Table 4.1. We are implicitly assuming that shocks to the market portfolio are perfectly correlated with the real SDF. Brennan et al. (2004) assume that $\bar{\eta} = 0.70$ while the sample Sharpe ratio on the market was 0.50 for the period 1952 to 2000. In Lettau and Wachter (2011), they find the Sharpe ratio on the market to be equal to 0.40 for period 1952 to 2004. We also set the long-run means of inflation's components to be equal to their unconditional means $(\bar{\pi}_c, \bar{\pi}_f, \bar{\pi}_e) = (3.78\%, 3.82\%, 4.43\%)$. We estimate the long-run mean of the instantaneous real interest rate $\hat{\bar{r}} = 1.10\%$ since the market inflation is estimated we are not able to identify the real rate sample moment before the Kalman filter estimation.

Our measurement equations consist of the set of nominal yield data, the inflation's components and the Sharpe ratio of the market. We let observed values to differ from true values by including a measurement term for all observed processes. The measurement term in the nominal yield data is assumed to be the same and proportional of the bond's maturity, $\sigma^2_{\varepsilon_\tau} = \sigma^2_b/\tau_j$, as in Brennan et al. (2004). The measurement term for inflation components are set equal to the variance of the residual of three univariate AR(1) processes. Besides, measurement errors for all variables are assumed to be serially and cross-sectionally uncorrelated. Estimates of the parameters in the transition equations (4.30) and the measurement equations (4.31)-(4.32) including the weights of market inflation's components are reported in Table 4.2 and Table 4.3.

Table 4.2: State factors estimates

We estimate the set of parameters for the state variables of the model by running the Kalman filter. In the transition equations we fix unconditional means of all state variables but the real interest rate to match their unconditional sample moments. Our measurement equations consist of the set of nominal yield data, the inflation's components and the Sharpe ratio on the market.

Real interest rate: $dr_t = \kappa_r[\bar{r} - r_t]dt + \sigma_r dZ_{r,t}$

	$\hat{\kappa}_r$	$\hat{\bar{r}}$	$\hat{\sigma}^2_r$
Value	1.507	1.104	4.211
Std. error	(0.147)	(2.855)	(0.537)

Sharpe ratio: $d\eta_t = \kappa_\eta[\bar{\eta} - \eta_t]dt + \sigma_\eta dZ_{\eta,t}$

	$\hat{\kappa}_\eta$	$\bar{\eta}$	$\hat{\sigma}^2_\eta$
Value	0.045	0.3942	0.947
Std. error	(0.126)	-	(0.289)

Inflation Components: $d\pi_{j,t} = \kappa_{\pi_j}[\bar{\pi}_j - \pi_{j,t}]dt + \sigma_{\pi_j} dZ_{\pi_j,t}$

	$\hat{\kappa}_{\pi_c}$	$\bar{\pi}_c$	$\hat{\sigma}^2_{\pi_c}$	$\hat{\kappa}_{\pi_f}$	$\bar{\pi}_f$	$\hat{\sigma}^2_{\pi_f}$	$\hat{\kappa}_{\pi_e}$	$\bar{\pi}_e$	$\hat{\sigma}^2_{\pi_e}$
Value	0.238	3.78	0.558	0.004	3.8229	0.738	0.415	4.4309	1.416
Std. error	(0.261)	-	(0.211)	(0.014)	-	(0.203)	(0.131)	-	(0.385)

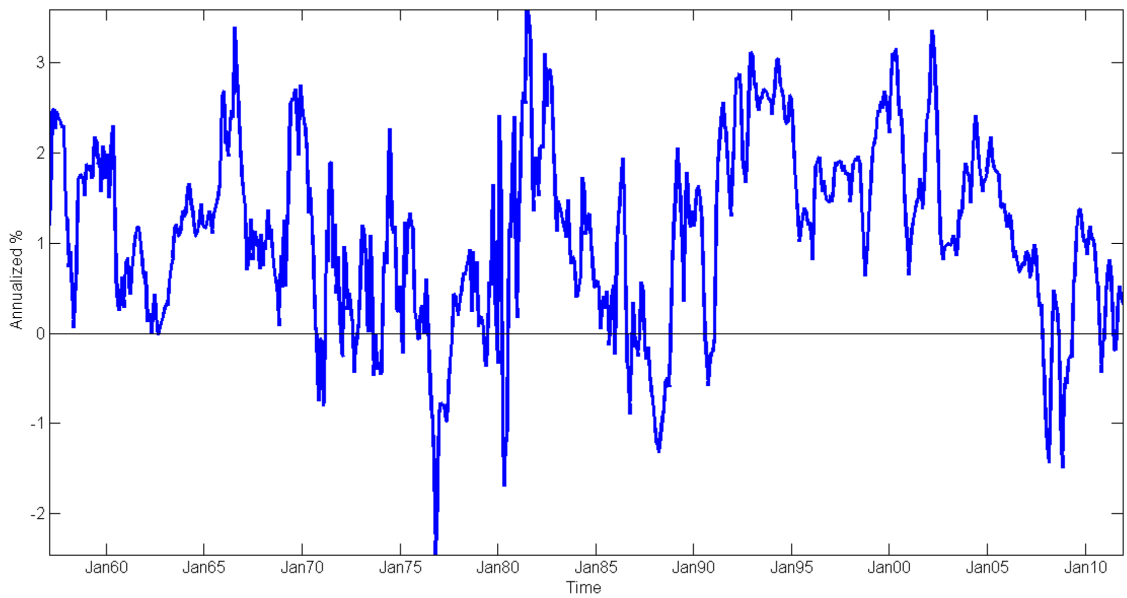
Inflation: $\pi_t = \omega_c\pi_{c,t} + \omega_e\pi_{e,t} + \omega_f\pi_{f,t}$ where $\frac{dQ_t}{Q_t} = \pi_t dt + \sigma_q dZ_{q,t}$

	$\hat{\omega}_c$	$\hat{\omega}_f$	$\hat{\omega}_e$	$\hat{\sigma}^2_q$
Value	0.551	0.173	0.275	0.01
Std. error	(0.023)	(0.003)	(0.273)	(0.497)

Table 4.3: Correlation estimates

The table shows estimates of the correlation between the state variables of the model. Top values refer to the point estimates whereas bottom values between parenthesis show standard error of the estimates.

Shock	$dZ_{r,t}$	$dZ_{\eta,t}$	$dZ_{q,t}$	$dZ_{\pi_c,t}$	$dZ_{\pi_f,t}$	$dZ_{\pi_e,t}$
$dZ_{m,t}$	-0.391	0.005	-0.321	-0.467	0.899	0.305
	(0.024)	(0.124)	(1.636)	(0.100)	(0.106)	(0.457)
$dZ_{r,t}$		0.604	-0.493	0.648	0.016	-0.359
		(0.000)	(22.857)	(0.334)	(0.000)	(0.111)
$dZ_{\eta,t}$			-0.532	0.316	-0.43	-0.046
			(6.969)	(0.001)	(0.102)	(0.001)
$dZ_{q,t}$				0	0	0
				-	-	-
$dZ_{\pi_c,t}$					0.2714	0.1268
					-	-
$dZ_{\pi_f,t}$						0.1027
						-

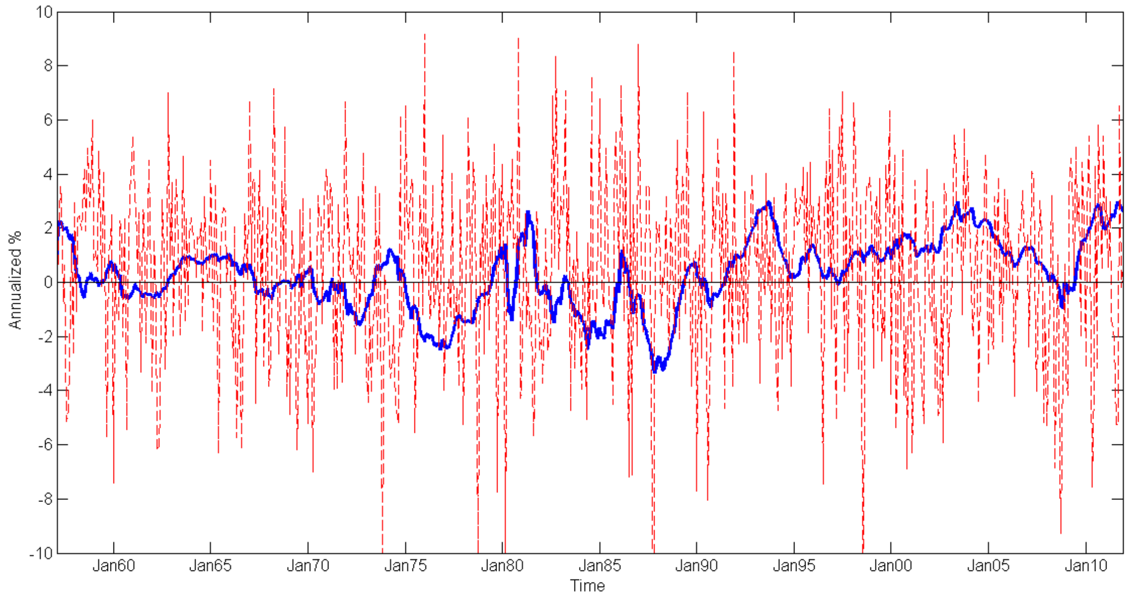


The figure shows the estimated instantaneous real interest rate from February 1957 to December 2011. The estimated real interest rate is filtered out by running the Kalman filter to the model.

Figure 4.1: Expected real interest rate

Figure 4.1 shows the estimated instantaneous real risk-free rate which presents a sample mean of 1.09% and positive values for much of the sample. The estimated mean-reverting parameter of the process towards the unconditional long-run mean is statistically significant and equal to $\kappa_r = 1.5$ implying a monthly autocorrelation of 0.88. The volatility of the shocks affecting the real interest rate is estimated to be around $\sigma_r = 2.05\%$ implying that r_t is a highly volatile and persistent process.

Figure 4.2 plots the time series for the realized and the estimated maximum Sharpe ratio of the economy. The estimated Sharpe ratio has an estimated mean-reverting parameter of $\kappa_\eta = 0.045$ and a volatility of 1.19%. Importantly, while shocks to the Sharpe ratio seem to be uncorrelated with the real SDF ($\rho_{m\eta} = 0.005, |t| = 0.039$), shocks to the real interest rate are significantly negative correlated with the real SDF ($\rho_{mr} = -0.391, |t| = 16.58$), implying that there is a risk premium associated with changes in the instantaneous real interest rate.

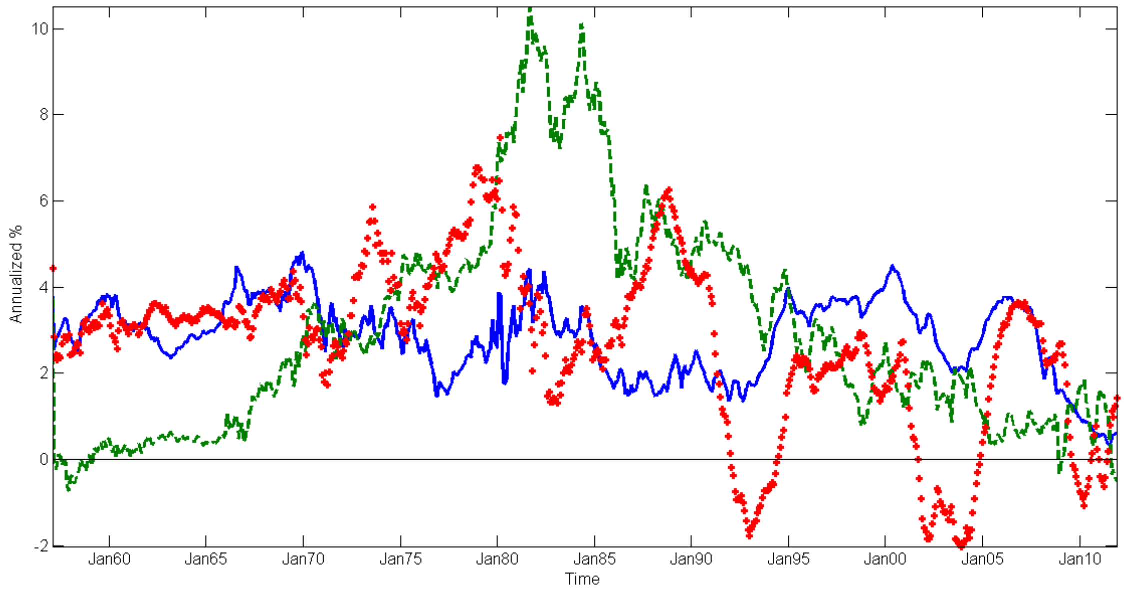


The figure plots the realized (red dashed) and the estimated (blue solid) Sharpe Ratio from February 1957 to December 2011. The realized Sharpe ratio is obtained after fitting an EGARCH model for the market excess return. The estimated Sharpe ratio is filtered out by running the Kalman filter.

Figure 4.2: The maximum Sharpe ratio of the economy

Figure 4.3 exhibits the time series of expected estimates of the market inflation components: core in the blue solid line, food in the green dashed line, and energy in the red dot line. The expected core inflation is the most stable component of inflation with an estimated annualized volatility of $\sigma_{\pi_c} = 0.75\%$. The mean of the estimated core process is around 2.83% while its range is between 0.33% and 4.81%. It reaches maximum values at the beginning of the Seventies and the Eighties, and in the second half of the

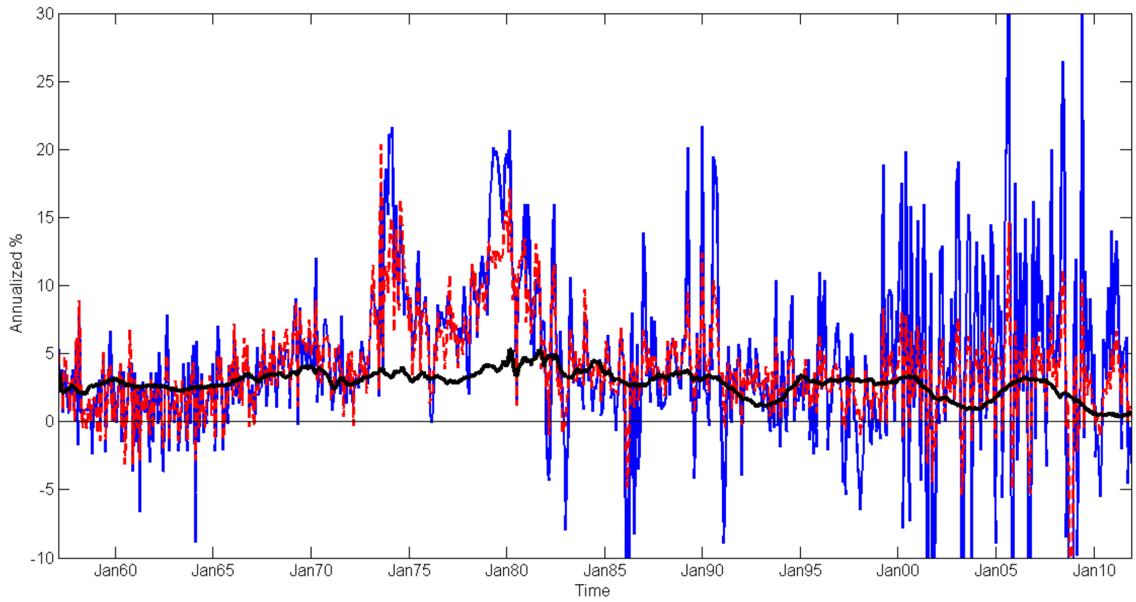
Nineties. The most volatile component of inflation is energy with an estimated annualized volatility of $\sigma_{\pi_e} = 1.41\%$ while the most persistent component is food with an estimated mean-reverting parameter $\kappa_{\pi_f} = 0.004$. Expected energy inflation hits maximum values during the 1973 oil crisis and the Gulf War. After then, the expected energy inflation is much more moderated and it raises again until the last financial crisis. All expected inflation components are correlated with shocks to the real SDF but the estimated coefficient for energy inflation is not significantly different from zero. This implies that the inflation risk premium will be associated with the weighted sum of inflation core and food premiums.



The figure plots estimates of the expected inflation components filter out by running the Kalman filter from February 1957 to December 2011: Core (blue solid), food (green dashed) and energy (red dot).

Figure 4.3: Market inflation's components

Figure 4.4 shows the time series of the expected market inflation (black solid) which is a weighted average of the expected core inflation ($\hat{\omega}_c = 0.55$), food inflation ($\hat{\omega}_f = 0.175$), and energy inflation ($\hat{\omega}_e = 0.275$). All weights significantly differ from the weights that composed the headline inflation (see Panel B.ii. in Table 4.1) except the energy component in which we cannot reject the null hypothesis because of the high standard error of its parameter estimate. The Figure also exhibits the realized headline inflation (red dashed) and the realized market inflation (blue solid) in which the last one presents a higher volatility, in particular in the last part of the sample. The estimated volatility of unexpected shocks to market inflation is around $\sigma_q = 0.1\%$ and by assumption shocks to the unexpected part are uncorrelated to shocks to expected inflation components.



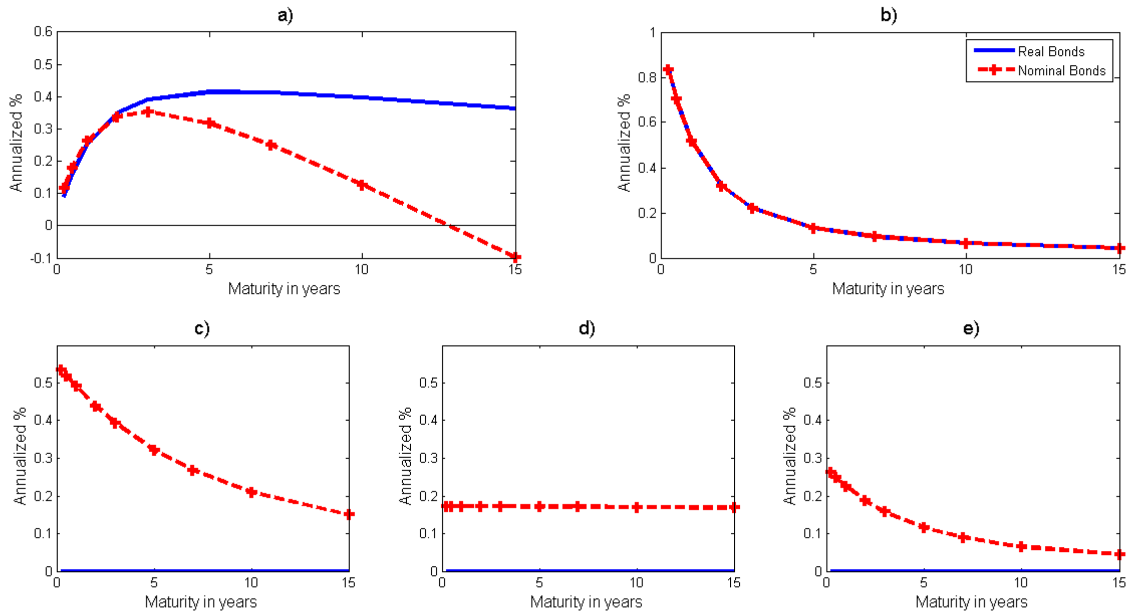
The figure plots the estimated expected market inflation in the black solid line from February 1957 to December 2011. The blue solid line is the realized market inflation while the red dashed represents the realized CPI-U inflation from the same period.

Figure 4.4: The market inflation

4.4.2 Bond yields

In equation (4.15) and (4.27) we obtain yields of real and nominal bonds as a function of the state variables of the model. Figure 4.5 plots the bonds' yields sensitivities to changes in the state variables: Plot a) the sensitivities of yields to the Sharpe ratio (top left); Plot b) the sensitivities of yields to the real riskless rate (top right); Plot c) the sensitivities of yields to the expected core inflation (bottom left); Plot d) the sensitivities of yields to the expected food inflation (bottom middle); and, Plot e) the sensitivities of yields to the expected energy inflation (bottom right). Blue solid lines denote the sensitivities of real yields to the state factors and red dashed lines denote the sensitivities of nominal yields.

Plot a) shows that an increase in the Sharpe ratio positively affects yields of real bonds for all maturities and the magnitude of these effects increases as a function of maturity, reaching a peak in bonds with maturity of five years and then decreasing at small rate for higher maturities. Yields of nominal bonds have a similar pattern but the highest sensitivity to changes in the Sharpe ratio is for nominal bonds with maturity around 3 years and then the magnitude of the effect decreases at a higher rate than for real bonds. Indeed, for nominal bonds with maturities higher than 13 years an increase in the price of risk reduces nominal yields of long-term bonds. The sensitivities of real and nominal yields to changes in the Sharpe ratio suggest that real bonds of all maturities and nominal bonds with short- and medium-maturities contain a positive premium over the riskless rate. However, long-term (more than 13 years) nominal bonds present a negative premium over the risk-free asset suggesting that long-term nominal bonds act as a hedge to changes in the investment opportunity set.



The figures plot the estimated sensitivities of bond yields to changes in the state variables: **a)** Sharpe ratio η_t; **b)** Real interest rate r_t; **c)** Core inflation $\pi_{c,t}$; **d)** Food inflation $\pi_{f,t}$; **e)** Energy inflation $\pi_{e,t}$.

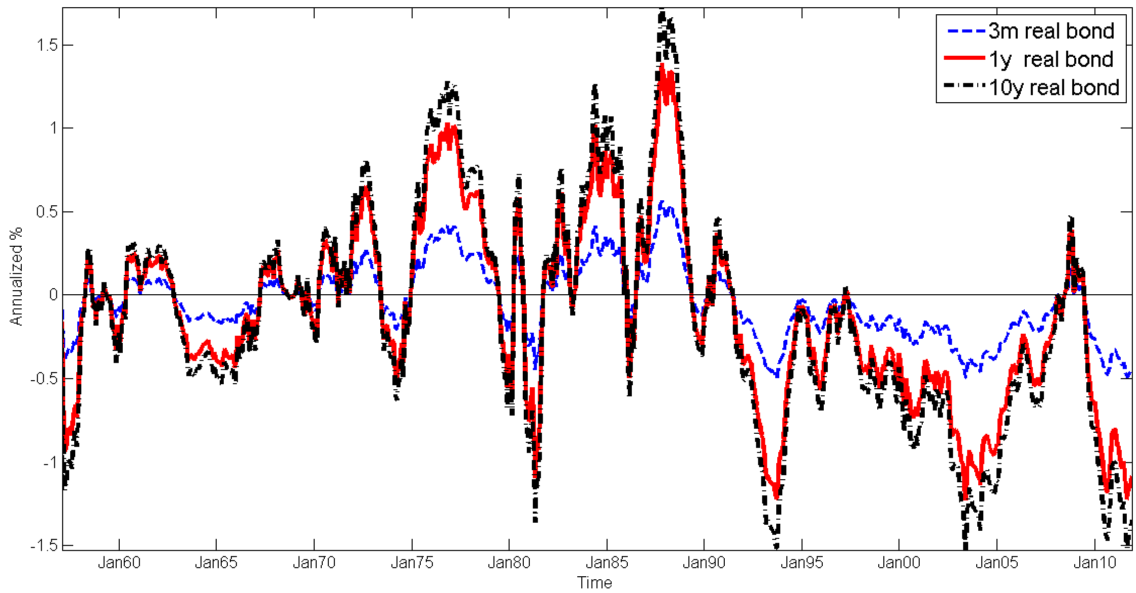
Figure 4.5: Sensitivities of bond yields

Plot b) in Figure 4.5 shows that an increase in the real risk-free rate increases real and nominal yields in the same magnitude and in both types of debt the impact is larger for short-term yields than for medium- and long-term maturities. The real riskless rate is assumed to follow an OrnsteinUhlenbeck process which is the continuous-time analogue of the discrete-time AR(1) process implying that over time the process tends to return to its long-term mean and the impact of shocks to the real risk-free rate dies out exponentially for long-term yields. The higher is the mean-reverting parameter the weaker is the sensitivities of long-term yields to the instantaneous real rate.

Meanwhile, inflation components only affects nominal yields and the magnitude of the effect depends on: i) the weight as a component in the market inflation, ω_j; and ii) its persistence, given by κ_{π_j}. Core inflation has the greatest effect on nominal yields and the impact is much higher for short-term yields than for medium- and long-term maturities, as it is shown in Plot c). For instance, a 1% increment in the expected core inflation increases 3m and 6m nominal yields in more than 50 basis points while the impact on the nominal yields with more than 7 years to mature is less than halved. For the case of the expected food inflation, its estimated mean-reverting parameter is close to zero implying that the process follows almost a random walk and its shocks does not die out in time. Then, a change in expected food inflation affects nominal yields for all maturities in a similar magnitude. Finally, shocks to expected energy inflation have a higher impact in short-term yields (around 26 basis points for the short-end of the yield curve per one percent increment in the expected energy inflation) with an effect that dies out exponentially for longer maturities.

4.4.3 Risk premiums

Bonds premiums over the real risk-free rate are obtained using equation (4.5) where the risk premium of any security depends on the price of risk, the volatility of the asset's return, and the correlation between the asset and the real SDF. The risk premiums on real zero-coupon bonds over the real rate are obtained in equation (4.16) in which they are determined by the covariance between the real SDF and the two sources of risk affecting real bonds yields: the Sharpe ratio and the instantaneous interest rate.



The figure exhibits the estimated risk premiums on real zero-coupon bonds over the real rate which are obtained in equation (4.16) from February 1957 to December 2011.

Figure 4.6: Real bond risk premium

Figure 4.6 shows the time-varying behaviour of the risk premiums on 3 months, 1 and 10 years real zero-coupon bonds which are (negatively) proportional to the maximum Sharpe ratio. The average value of the estimated risk premium on a 10 year real bonds is around -15 basis points reaching a maximum of 170 basis points at the end of the

Eighties, and a minimum values of almost -150 basis points. The estimated correlation coefficient between the SDF and the instantaneous real interest rate is significant and negative ($\rho_{mr} = -0.391$) implying that when the Sharpe ratio is positive, zero-coupon real bonds carry negative premiums and they behave as hedger to the real economy conditions.[15] On the other hand, the estimated correlation coefficient between the SDF and the Sharpe ratio is not significantly different from zero implying that there is no risk premium associated with this factor.

For the case of nominal bonds, their risk premiums are obtained in equation (4.28) in which the last term, $-\sigma_q \rho_{mq} \eta_t$, represents the instantaneous inflation risk premium (IIRP) which is indistinguishable different from zero getting maximum values of $9bp$ and minimum of $-10bp$. Table 4.3 shows that we cannot reject the null hypothesis that ρ_{mq}, $\rho_{q\eta}$, and ρ_{qr} differ from zero implying that shocks to the unexpected market inflation appear to be uncorrelated with the real SDF, the maximum Sharpe ratio, and the real risk-free rate. Besides, we assume that shocks to the expected inflation's components are uncorrelated with shocks to the unexpected market inflation for which $\rho_{q\pi_j}$ are all equal to zero. Then, the nominal bonds' risk premium are determined by the covariance between risk factors affecting nominal bonds (Sharpe ratio, real interest rate, inflation's components) and the real SDF.

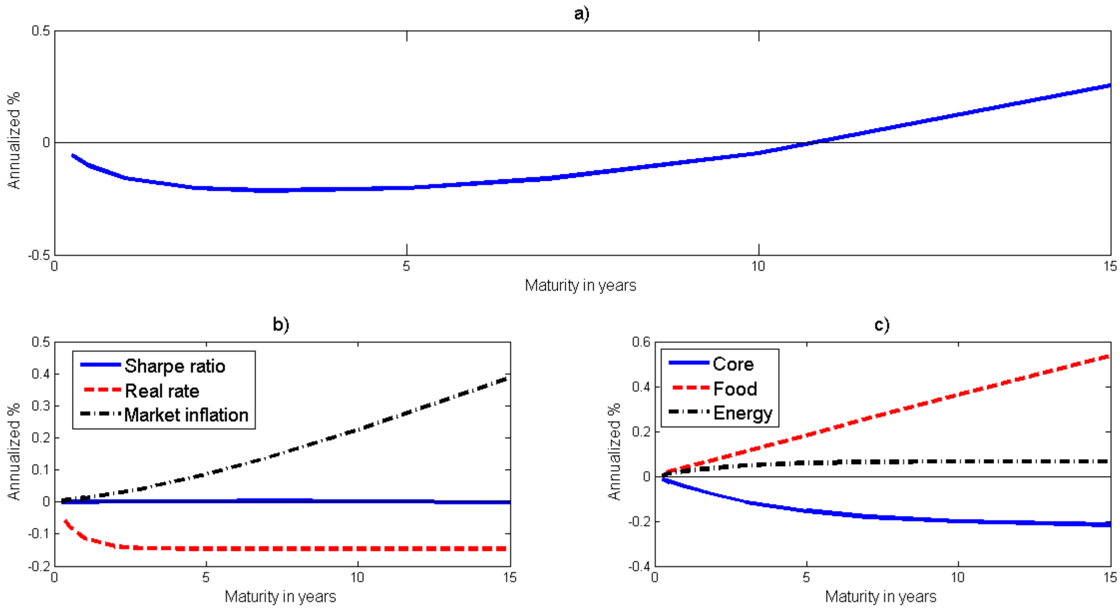
Plot a) in Figure 4.7 exhibits the average risk premium curve on nominal zero-coupon bonds over the real riskless rate. The plot shows a hump-shaped risk premium curve with negative values for short- and medium-term maturities and positive risk premium for long-term nominal bonds. The minimum value of $-21.5bp$ belongs to a 3 year nominal bond while long-term bonds (more than 10 years) present a positive increasing risk premium.

Plot b) in the same Figure shows the decomposition of the nominal risk premium in the tree main factors affecting nominal yields (real rate, Sharpe ratio, and market inflation). The red dashed curve denotes the average real rate risk premium component which is negative for all maturities and flat for medium and long-term bonds. The real rate risk premium component for 2 year nominal bond is around $-14bp$ while it is almost $-15bp$ for 15 year nominal bond. The black dot-dashed curve denotes the average market inflation risk premium on nominal bonds which is characterized by and upward sloping shape. The range of the market inflation risk premium is between $1bp$ for a 1 year nominal bond and $40b$ for 15 year nominal bond.

Plot c) exposes the decomposition of the market inflation risk premium in the tree components: core, food and energy inflation. Interestingly, core inflation presents a negative downward sloping risk premium curve which flattens out as one moves to longer maturities with an minimum of $-21bp$ for 15 year nominal bond. That is, the longer the maturity, the lower the core inflation risk premium, with diminishing marginal decreases. Energy inflation although it is the most volatile component of market inflation does not present a significant risk premium since it seems to be uncorrelated with the real SDF. On

[15]Brennan et al. (2004) find that that recessions are usually associated with an increasing Sharpe ratio.

the other hand, the most persistence component, food inflation, shows a positive upward sloping risk premium curve reaching more than $35bp$ for nominal bonds with maturities longer than 10 years.

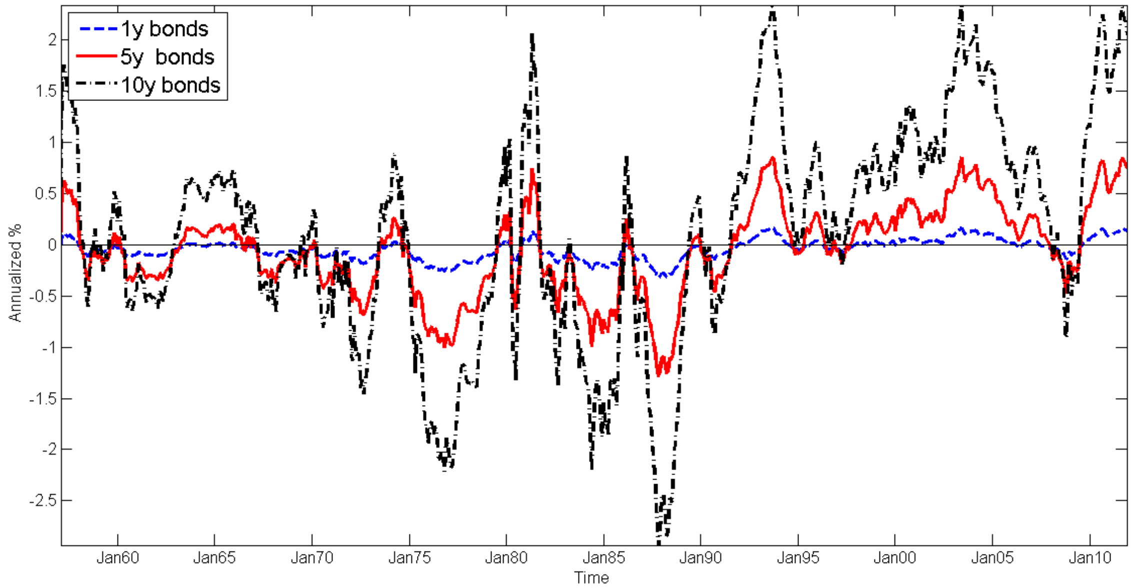


Plot a) exhibits the estimated average risk premium curve on nominal zero-coupon bonds over the real riskless rate for February 1957 to December 2011. Plot b) shows the decomposition of the nominal risk premium in the tree main factors affecting nominal yields (real rate, Sharpe ratio, and market inflation). Plot c) exposes the decomposition of the market inflation risk premium in the tree components: core, food and energy inflation.

Figure 4.7: Nominal bond risk premium

From the Treasury point of view, by issuing real bonds instead of nominal ones a government may reduce borrowing costs by not having to pay the inflation risk premium. Roush (2008) and Fleckenstein et al. (2010), among others have shown that the TIPS program started in January 1997 have been costly for the U.S. Treasury. In other words, the Treasury have increased the financing cost by issuing TIPS as compared with the conventional nominal bonds.

Figure 4.8 plots the risk premium of nominal bonds over real bonds with the same maturity obtained in equation (4.29) for bonds with maturity of 1, 5 and 10 years. The Figure shows that the usual argument supporting the issuance of real bonds is no completely true. There are periods in which the risk premium included in real bonds is higher than for nominal bonds with the same maturity. The difference in premium between nominal and real bonds with the same maturity is not only time-varying but also the sign changes through time. For instance, the Figure shows that during the beginning of the last financial crisis the difference in the risk premiums was around $100bp$. That is, the Treasury by issuing a 10 year real bonds in that period instead of nominal bonds with the same maturity have increased their financial cost in 1% per every dollar of the new real debt issued.



The figure shows the estimated nominal bond risk premium over a real bond with the same maturity for bonds with maturity of 1, 5 and 10 years from February 1957 to December 2011.

Figure 4.8: Nominal maturity risk premium

4.5 Concluding Remarks

A fundamental question that arises when studying the benefits of issuing IL bonds is whether the to which IL bonds are linked accurately reflects inflation exposure of the investors, which we define as the "market" inflation. The goal of this chapter is to address whether the U.S. expected market inflation differs from the inflation index used for TIPS and determine the main implications for the benefits of issuing IL bonds.

For this purpose, we develop and estimate a no-arbitrage term structure model that fits U.S. nominal bonds data in order to obtain a measure of the U.S. expected market inflation as the weighted average of the main component of the CPI-U inflation. The main feature of the model is that it allows for time-varying risk premiums by assuming a stochastic variation in the real investment opportunities which is given by the time variation of the real interest rate and the maximum Sharpe ratio of the economy. To estimate the market inflation, we introduce an exogenous process for the price level such that the expected market inflation is a weighted (unknown) average of expected core, food, and energy inflation.

We estimate the U.S. market inflation as the weighted average of the expected core, food inflation, and energy inflation in which all weights significantly differ from the weights of the CPI-U (known as headline) inflation, except the energy component in which we can not reject the null hypothesis because of the high standard error of its parameter estimate. This result implies an inflation basis risk in the TIPS' indexation rule due to the fact that market inflation differs from the inflation index to which TIPS are linked.

We find time-varying risk premiums on real and nominal zero-coupon bonds over the real riskless rate. We also find that the risk premiums on nominal bonds over real bonds with the same maturity are not only time-varying but also their signs change through time which poses doubts about the general agreement that by issuing IL bonds instead of nominal ones a government will reduce borrowing cost by not having to pay the inflation risk premium.

Chapter 5

Concluding Remarks

5.1 Summary of Conclusions and Future Research

Over the last few decades, a large number of articles by academics and practitioners have examined the arguments for and against issuing inflation-linked (IL) bonds. In general, the issuance of IL bonds is justified on welfare gains since it is argued that they provide benefits to the society. This thesis is devoted to studying the benefits of IL bonds, specifically to account for the benefits that Treasury Inflation-Protected Securities (TIPS) have provided to the different stakeholders of the economy.

In Chapter 2 we solve an optimal portfolio choice problem to measure the benefits of TIPS to investors concerned with maximizing real wealth. We show how the introduction of a real riskless asset completes the investor asset space, by contrasting optimal portfolio allocations with and without such assets. We use historical data to quantify gains from the availability of TIPS in the presence of other asset classes such as equities, commodities, and real estate. We draw a distinction between buy-and-hold long-term investors for whom TIPS fully displace nominal risk-free assets and short-term investors for whom TIPS improve the investment opportunity set of real returns. Finally, we show how gains from TIPS are tempered by availability of alternative assets that covary with inflation, such as gold and real estate. The future research agenda includes extensions to our simple model where for example one could consider dynamic rebalancing of the portfolio as well as including other instruments in the investor's opportunity set such as inflation derivatives. A model with intermediate rebalancing would allow to consider the benefits that TIPS provide to investors from a real rate hedging perspective, something which is missing in the model presented in Chapter 2.

In Chapter 3 we develop a theoretical term structure model in a preferred-habitat framework to study the implications about the benefits of issuing IL bonds instead of conventional nominal ones. Regarding the benefits of IL bonds the main findings are: i) that the inflation risk premium included in nominal bonds' yields can be negative which implies that IL bonds can provide a more expensive source of funding than conventional nominal bonds; and, ii) that in periods of financial distress the difference between nominal and real yields may not adequately capture the compensation that buy-and-hold investors demand to cover expected inflation. An extension to this Chapter is to fit data to the preferred-habitat model. Alternatively, we believe that testing the implications predicted by theoretical model presented in Chapter 3 is worthy for future research.

Finally, in Chapter 4 we develop and estimate a no-arbitrage term structure model. We fit U.S. nominal bonds data in order to obtain a measure of the U.S. expected market inflation as the weighted average of the main component of the CPI-U inflation. The results are that market inflation differs from the inflation measured by the CPI-U which implies that there exists an inflation basis risk in TIPS due to their indexation rule. We also find that the difference in premium between nominal and real bonds with the same maturity is not only time-varying but also the sign changes through time. This implies that a government would not necessarily save the inflation risk premium by issuing IL bonds instead of nominal ones. Ongoing research on this problem includes improvements in the robustness of our estimation methodology.

Appendix A

Appendix: The U.S. Breakeven Inflation Rates in a Preferred-Habitat Model

A.1 US Auction Investor Class Descriptions

The investor class is categorized in eight groups:

1. **Federal Reserve System:** Includes the Federal Reserve Banks System Open Market Account (SOMA).
2. **Depository Institutions:** Includes banks, savings and loan associations, credit unions, and commercial bank investment accounts.
3. **Individuals:** Includes individuals, partnerships, personal trusts, estates, non-profit and tax-exempt organizations, and foundations.
4. **Dealers and Brokers:** Includes primary dealers, other commercial bank dealer departments, and other non-bank dealers and brokers.
5. **Pension and Retirement Funds and Insurance Companies:** Includes pension and retirement funds, state & local pension funds, life insurance companies, casualty and liability insurance companies, and other insurance companies.
6. **Investment Funds:** Includes mutual funds, money market funds, hedge funds, money managers, and investment advisors.
7. **Foreign and International:** Includes private foreign entities, non-private foreign entities placing tenders external of the Federal Reserve Bank of New York (FRBNY), and official foreign entities placing tenders through FRBNY.

8. **Other:** For coupons– "Other" represents the residual from categories not specified in investor class descriptions above.

A.2 Proofs of the Model

A.2.1 Proof of Lemma 3.3.1 (Arbitrageurs' first order condition)

Substituting (3.8a) and (3.8b) into (3.4) we re-express the arbitrageurs' optimization problem as

$$\max_{\{x_{t,\tau},x^{\$}_{t,\tau}\}_{\tau\in(0,T]}} \mathrm{E}_t\left[\frac{dW_t}{W_t}\right]-\frac{\gamma}{2}\mathrm{Var}_t\left[\frac{dW_t}{W_t}\right] \tag{A.1}$$

$$\begin{aligned}\frac{dW_t}{W_t}=&\left[r_t+\int_0^T x_{t,\tau}\Big[\mu_{t,\tau}-r_t\Big]d\tau+\int_0^T x^{\$}_{t,\tau}\Big[\mu^{\$}_{t,\tau}-\pi_t-r_t\Big]d\tau\right]dt\\&-\left[\int_0^T\Big[x_{t,\tau}B(\tau)+x^{\$}_{t,\tau}B_{\$}(\tau)\Big]d\tau\right]\sigma_r dZ_{r,t}-\left[\int_0^T x^{\$}_{t,\tau}C_{\$}(\tau)d\tau\right]\sigma_\pi dZ_{\pi,t}.\end{aligned} \tag{A.2}$$

The expected value and variance of the return of the real wealth is given by

$$\mathrm{E}_t\big[dW_t/W_t\big]=r_t+\int_0^T x_{t,\tau}\big[\mu_{t,\tau}-r_t\big]d\tau+\int_0^T x^{\$}_{t,\tau}\big[\mu^{\$}_{t,\tau}-\pi_t-r_t\big]d\tau, \tag{A.3}$$

$$\mathrm{Var}_t\big[dW_t/W_t\big]=\sigma_r^2M_r^2+\sigma_\pi^2M_\pi^2+2\rho_{r,\pi}\sigma_r\sigma_\pi M_rM_\pi, \tag{A.4}$$

where

$$M_r=\int_0^T\Big[x_{t,\tau}B(\tau)+x^{\$}_{t,\tau}B_{\$}(\tau)\Big]d\tau, \tag{A.5}$$

$$M_\pi=\int_0^T x^{\$}_{t,\tau}C_{\$}(\tau)d\tau. \tag{A.6}$$

The arbitrageurs' optimization problem is

$$\begin{aligned}\max_{\{x_{t,\tau},x^{\$}_{t,\tau}\}_{\tau\in(0,T]}}&\int_0^T x_{t,\tau}\Big[\mu_{t,\tau}-r_t\Big]d\tau+\int_0^T x^{\$}_{t,\tau}\Big[\mu^{\$}_{t,\tau}-\pi_t-r_t\Big]d\tau\\&-\frac{\gamma}{2}\left[\sigma_r^2M_r^2+\sigma_\pi^2M_\pi^2+2\rho_{r,\pi}\sigma_r\sigma_\pi M_rM_\pi\right].\end{aligned} \tag{A.7}$$

Point-wise maximization (A.7) of leads to (3.10a) and (3.10b).

A.2.2 Bond Yields

We re-write the market-clearing constraints with (3.5), and taking into account (3.1a) and (3.7a):

$$s_\tau = x_{t,\tau} + \alpha\Big[A(\tau) + B(\tau)r_t - \tau\beta\Big], \tag{A.8}$$

$$s^{\$}_\tau = x^{\$}_{t,\tau}. \tag{A.9}$$

Then, we re-write the arbitrageurs' first order condition taking into account (3.9a), (3.9b) and replacing $x_{t,\tau}$ and $x^{\$}_{t,\tau}$ with (A.8) and (A.9). The first order condition of (3.3) with respect to the proportion invested in IL bonds is given by

$$A'(\tau) + B'(\tau)r_t - B(\tau)\kappa_r\big[\bar{r} - r_t\big] + \frac{1}{2}B(\tau)^2\sigma_r^2 - r_t\Big[1 - B(\tau)\gamma\sigma_r^2\alpha\int_0^T B(\tau)^2 d\tau\Big] = B(\tau)\gamma\sigma_r\Big[\sigma_r N_r + \rho_{r,\pi}\sigma_\pi N_\pi\Big], \tag{A.10}$$

and the first order condition of (3.3) with respect to the proportion invested in nominal bonds is given by

$$\begin{aligned} &A'_{\$}(\tau) + B'_{\$}(\tau)r_t + C'_{\$}(\tau)\pi_t - B_{\$}(\tau)\kappa_r\big[\bar{r} - r_t\big] - C_{\$}(\tau)\kappa_\pi\big[\bar{\pi} - \pi_t\big] - \pi_t - r_t \\ &+ \frac{1}{2}\Big[B_{\$}(\tau)^2\sigma_r^2 + C_{\$}(\tau)^2\sigma_\pi^2 + 2\rho_{r,\pi}\sigma_r\sigma_\pi B_{\$}(\tau)C_{\$}(\tau)\Big] \\ &+ r_t\Big[\Big[B_{\$}(\tau)\gamma\sigma_r^2 + C_{\$}(\tau)\gamma\rho_{r,\pi}\sigma_r\sigma_\pi\Big]\int_0^T \alpha B(\tau)^2 d\tau\Big] \\ &= B_{\$}(\tau)\gamma\sigma_r\Big[\sigma_r N_r + \rho_{r,\pi}\sigma_\pi N_\pi\Big] + C_{\$}(\tau)\gamma\sigma_\pi\Big[\sigma_\pi N_\pi + \rho_{r,\pi}\sigma_r N_r\Big], \end{aligned} \tag{A.11}$$

where

$$N_r = \int_0^T \alpha B(\tau)\Big[\tau\beta - A(\tau)\Big]d\tau + \int_0^T \Big[s_\tau B(\tau) + s^{\$}_\tau B_{\$}(\tau)\Big]d\tau, \tag{A.12}$$

$$N_\pi = \int_0^T s^{\$}_\tau C_{\$}(\tau)d\tau. \tag{A.13}$$

Proof of Proposition 3.4.1 (Real yield of an IL bond)

Equation (A.10) is an affine equation in the short-term real interest rate, r_t. Setting linear terms in r_t to zero we get

$$B'(\tau) + B(\tau)\left[\kappa_r + \gamma\sigma_r^2 \int_0^T \alpha B(\tau)^2 d\tau\right] - 1 = 0, \tag{A.14}$$

which is a linear differential equation in $B(\tau)$ where the solution is given by (3.12b) and (3.13a). Setting constant terms of (A.10) to zero we find

$$A'(\tau) - B(\tau)\kappa_r\bar{r} + \frac{1}{2}B(\tau)^2\sigma_r^2 - B(\tau)\gamma\sigma_r\left[\sigma_r N_r + \rho_{r,\pi}\sigma_\pi N_\pi\right] = 0. \tag{A.15}$$

We can now integrate (A.15) taking into account the boundary conditions $A(0) = B(0) = 0$ to solve the function $A(\tau)$:

$$\int_t^T A'(T-s)ds = \int_t^T \left[B(T-s)\kappa_r\bar{r} - \frac{1}{2}B(T-s)^2\sigma_r^2 + B(T-s)\gamma\sigma_r\left[\sigma_r N_r + \rho_{r,\pi}\sigma_\pi N_\pi\right]\right]ds$$

$$A(\tau) = z_r \int_0^\tau B(u)du - \frac{\sigma_r^2}{2}\int_0^\tau B(u)^2 du. \tag{A.16}$$

where

$$z_r = \kappa_r\bar{r} + \gamma\sigma_r^2 N_r + \gamma\rho_{r,\pi}\sigma_r\sigma_\pi N_\pi. \tag{A.17}$$

Note that N_r contains $A(\tau)$, thus we replace (A.16) and (A.12) into (A.17) and get

$$z_r = \frac{\kappa_r\bar{r} + \gamma\sigma_r^2\beta\int_0^T \tau\alpha B(\tau)\, d\tau + \frac{\gamma\sigma_r^4}{2}\int_0^T \alpha B(\tau)\left[\int_0^\tau B(u)^2du\right] d\tau + \gamma\sigma_r Q_r}{1 + \gamma\sigma_r^2\int_0^T \alpha B(\tau)\left[\int_0^\tau B(u)du\right]d\tau}, \tag{A.18}$$

where

$$Q_r = \sigma_r \int_0^T \left[s_\tau B(\tau) + s_\tau^\$ B_\$(\tau)\right] d\tau + \rho_{r,\pi}\sigma_\pi \int_0^T s_\tau^\$ C_{,\$}(\tau)\, d\tau. \tag{A.19}$$

Taking into account (3.13a) and $\kappa_r^* \int_0^\tau B(u)du = \tau - B(\tau)$, we re-express κ_r^* as

$$\kappa_r^* = \frac{\kappa_r + \gamma\sigma_r^2\int_0^T \tau\alpha B(\tau)d\tau}{1 + \gamma\sigma_r^2\int_0^T \alpha B(\tau)\left[\int_0^\tau B(u)du\right]d\tau}. \tag{A.20}$$

Then (3.12a) holds since

$$\kappa_r^*\bar{r}^* = z_r. \tag{A.21}$$

Proof the dynamics of r_t under the risk-neutral measure

Re-writing (A.10) as

$$\begin{aligned} A'(\tau) + B'(\tau)r_t - B(\tau)\kappa_r\big[\bar{r} - r_t\big] + \frac{1}{2}B(\tau)^2\sigma_r^2 - r_t \\ = B(\tau)\gamma\sigma_r\Bigg[\sigma_r N_r + \rho_{r,\pi}\sigma_\pi N_\pi\Bigg] - B(\tau)r_t\gamma\sigma_r^2\int_0^T \alpha B(\tau)^2 d\tau, \quad \text{(A.22)} \end{aligned}$$

and taking into account (3.13a), (A.17) and (A.21), we get that

$$A'(\tau) + B'(\tau)r_t - B(\tau)\kappa_r^*\big[\bar{r}^* - r_t\big] + \frac{1}{2}B(\tau)^2\sigma_r^2 - r_t = 0. \quad \text{(A.23)}$$

Then, the dynamics of the short rate under the risk-neutral measure is characterized by the parameters $(\bar{r}^*, \kappa_r^*)$.

Proof of Proposition 3.4.2 (Nominal yield of a nominal bond)

Equation (A.11) is an affine equation in the short-term real interest rate and inflation. Setting linear terms in r_t and π_t to zero we get a system of linear differential equations in $B_\$(\tau)$ and $C_\$(\tau)$

$$\begin{bmatrix} B_\$'(\tau) \\ C_\$'(\tau) \end{bmatrix} + F \begin{bmatrix} B_\$(\tau) \\ C_\$(\tau) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \text{(A.24)}$$

where

$$F = \begin{bmatrix} \kappa_r + \gamma\sigma_r^2\int_0^T \alpha B(\tau)^2 d\tau & \gamma\rho_{r,\pi}\sigma_r\sigma_\pi\int_0^T \alpha B(\tau)^2 d\tau \\ 0 & \kappa_\pi \end{bmatrix}. \quad \text{(A.25)}$$

The solution to the system (A.24) is given in (3.17b) and (3.17c). Setting constant terms of (A.11) to zero we find

$$\begin{aligned} A_\$'(\tau) - B_\$(\tau)\kappa_r\bar{r} - C_\$(\tau)\kappa_\pi\bar{\pi} + \frac{1}{2}\Bigg[B_\$(\tau)^2\sigma_r^2 + C_\$(\tau)^2\sigma_\pi^2 + 2\rho_{r,\pi}\sigma_r\sigma_\pi B_\$(\tau)C_\$(\tau)\Bigg] \\ - B_\$(\tau)\gamma\sigma_r\Bigg[\sigma_r N_r + \rho_{r,\pi}\sigma_\pi N_\pi\Bigg] - C_\$(\tau)\gamma\sigma_\pi\Bigg[\sigma_\pi N_\pi + \rho_{r,\pi}\sigma_r N_r\Bigg] = 0. \quad \text{(A.26)} \end{aligned}$$

Integrating (A.26) and taking into account the boundary conditions we get the function $A_{\$}(\tau)$:

$$\begin{aligned} A_{\$}(\tau) = &\, z_r \int_0^\tau B_{\$}(u)du + z_\pi \int_0^\tau C_{\$}(u)du \\ & - \frac{1}{2}\left[\sigma_r^2 \int_0^\tau B_{\$}(u)^2 du + \sigma_\pi^2 \int_0^\tau C_{\$}(u)^2 du + 2\rho_{r,\pi}\sigma_r\sigma_\pi \int_0^\tau B_{\$}(u)C_{\$}(u)du\right], \end{aligned} \tag{A.27}$$

where

$$z_r = \kappa_r \bar{r} + \gamma\sigma_r^2 N_r + \gamma\rho_{r,\pi}\sigma_r\sigma_\pi N_\pi, \tag{A.28}$$

$$z_\pi = \kappa_\pi \bar{\pi} + \gamma\rho_{r,\pi}\sigma_\pi\sigma_r N_r + \gamma\sigma_\pi^2 N_\pi. \tag{A.29}$$

A.2.3 Market Prices of Risk

Proof of Proposition 3.4.6 (Market price of real interest rate risk and supply of IL bonds)

Let's compute $\frac{\partial A(\tau)}{\partial s_\tau}$ from (3.12)-(3.13)

$$\frac{\partial A(\tau)}{\partial s_\tau} = \frac{\gamma\sigma_r^2 B(\tau)\int_0^\tau B(u)du}{1+\gamma\sigma_r^2\int_0^T \alpha B(\tau)\Big[\int_0^\tau B(u)du\Big]d\tau}, \tag{A.30}$$

and replace it into equation (3.32)

$$\begin{aligned} \frac{\partial \lambda_{r,t}}{\partial s_\tau} &= \gamma\sigma_r^2\left[B(\tau) - \frac{\gamma\sigma_r^2\alpha\int_0^T B(\tau)^2\Big[\int_0^\tau B(u)du\Big]d\tau}{1+\gamma\sigma_r^2\alpha\int_0^T B(\tau)\Big[\int_0^\tau B(u)du\Big]d\tau}\right], \\ &= \gamma\sigma_r^2\left[\tau - \kappa_r^*\int_0^\tau B(u)du - \frac{\gamma\sigma_r^2\alpha\int_0^T B(\tau)^2\Big[\int_0^\tau B(u)du\Big]d\tau}{1+\gamma\sigma_r^2\alpha\int_0^T B(\tau)\Big[\int_0^\tau B(u)du\Big]d\tau}\right]. \end{aligned} \tag{A.31}$$

By definition α is a positive constant, then

$$\lim_{\alpha\to 0}\frac{\partial \lambda_{r,t}}{\partial s_\tau} = \gamma\sigma_r^2\frac{1-e^{-\kappa_r\tau}}{\kappa_r} \geq 0 \tag{A.32}$$

and

$$\lim_{\alpha\to\infty}\frac{\partial \lambda_{r,t}}{\partial s_\tau} = \lim_{\alpha\to\infty}\gamma\sigma_r^2\frac{1-e^{-\kappa_r^*\tau}}{\kappa_r^*} = 0, \tag{A.33}$$

since both $B(\tau)$ and $B(\tau)^2$ converge to zero when α goes to infinity but the latter does at a faster rate.

Proof of Proposition 3.4.7 (Market price of real interest rate risk and the real return of alternative investment opportunities)

Let's compute $\frac{\partial A(\tau)}{\partial \beta}$ from (3.12)-(3.13) and taking into account (A.20)

$$\begin{aligned}
\frac{\partial A(\tau)}{\partial \beta} &= \frac{\gamma\sigma_r^2 \int_0^T \tau\alpha B(\tau)d\tau \int_0^\tau B(u)du}{1+\gamma\sigma_r^2 \int_0^T \alpha B(\tau)\Big[\int_0^\tau B(u)du\Big]d\tau}, \\
&= \kappa_r^* \int_0^\tau B(u)du - \frac{\kappa_r \int_0^\tau B(u)du}{1+\gamma\sigma_r^2\alpha \int_0^T B(\tau)\Big(\int_0^\tau B(u)du\Big)d\tau}, \\
&= \tau - B(\tau) - \frac{\kappa_r \int_0^\tau B(u)du}{1+\gamma\sigma_r^2\alpha \int_0^T B(\tau)\Big(\int_0^\tau B(u)du\Big)d\tau}.
\end{aligned} \tag{A.34}$$

Replacing (A.34) into equation (3.33) we get

$$\begin{aligned}
\frac{\partial \lambda_{r,t}}{\partial \beta} &= \gamma\sigma_r^2\alpha \int_0^T \left[B(\tau) + \frac{\kappa_r \int_0^\tau B(u)du}{1+\gamma\sigma_r^2\alpha \int_0^T B(\tau)\Big[\int_0^\tau B(u)du\Big]d\tau}\right] B(\tau)\, d\tau, \\
&= \kappa_r^* - \kappa_r + \kappa_r \frac{\gamma\sigma_r^2\alpha \int_0^T B(\tau)\Big[\int_0^\tau B(u)du\Big]\, d\tau}{1+\gamma\sigma_r^2\alpha \int_0^T B(\tau)\Big[\int_0^\tau B(u)du\Big]d\tau}, \\
&= \kappa_r^* - \frac{\kappa_r}{1+\gamma\sigma_r^2\alpha \int_0^T B(\tau)\Big[\int_0^\tau B(u)du\Big]d\tau} \geq 0,
\end{aligned} \tag{A.35}$$

since

$$\lim_{\alpha\to 0} \frac{\partial \lambda_{r,t}}{\partial \bar{\beta}} = 0, \tag{A.36}$$

and

$$\lim_{\alpha\to \infty} \frac{\partial \lambda_{r,t}}{\partial \beta} = \kappa_r^*. \tag{A.37}$$

A.3 The Two-Factor Vasicek Model

Let derive a two-factor affine model where we assume the instantaneous real short-term interest rate, r_t, and the instantaneous expected rate of inflation, π_t, follow two correlated Ornstein-Uhlenbeck processes

$$dr_t = \kappa_r\big[\bar{r} - r_t\big]dt + \sigma_r dZ_{r,t}, \tag{A.38}$$

$$d\pi_t = \kappa_\pi\big[\bar{\pi} - \pi_t\big]dt + \sigma_\pi dZ_{\pi,t}, \tag{A.39}$$

where $(\bar{r}, \kappa_r, \sigma_r, \bar{\pi}, \kappa_\pi, \sigma_\pi)$ are positive constants and $(Z_{r,t}, Z_{\pi,t})$ are two correlated standard Brownian motions with $dZ_{r,t}dZ_{\pi,t} = \rho_{r,\pi}dt$.

Let the price of a zero-coupon nominal bond with a defined term $\tau = T - t$ be a function of the two underlying risk factors:

$$P^{\$}_{t,\tau} = P^{\$}(t, \tau, r_t, \pi_t). \tag{A.40}$$

We can determine the differential dynamics of the zero-coupon bond price by Itô's lemma:

$$\begin{aligned} dP^{\$}_{t,\tau} = {} & \frac{\partial P^{\$}_{t,\tau}}{\partial t}dt + \frac{\partial P^{\$}_{t,\tau}}{\partial r_t}dr_t + \frac{\partial P^{\$}_{t,\tau}}{\partial \pi_t}d\pi_t \\ & + \frac{1}{2}\left[\frac{\partial^2 P^{\$}_{t,\tau}}{\partial r_t^2}d\langle r\rangle(t) + \frac{\partial^2 P^{\$}_{t,\tau}}{\partial \pi_t^2}d\langle \pi\rangle(t) + 2\frac{\partial^2 P^{\$}_{t,\tau}}{\partial r_t \partial \pi_t}d\langle r, \pi\rangle(t)\right], \end{aligned} \tag{A.41}$$

where $d\langle . \rangle(t)$ and $d\langle r, \pi\rangle(t)$ represent the quadratic variation and covariation between r_t and π_t, respectively. Using (A.38)-(A.39), (A.41) can be stated as

$$dP^{\$}_{t,\tau} = P^{\$}_{t,\tau}\mu(\tau)dt + \frac{\partial P^{\$}_{t,\tau}}{\partial r_t}\sigma_r dZ_{r,t} + \frac{\partial P^{\$}_{t,\tau}}{\partial \pi_t}\sigma_\pi dZ_{\pi,t}, \tag{A.42}$$

where

$$\begin{aligned} P^{\$}_{t,\tau}\mu(\tau) = {} & \frac{\partial P^{\$}_{t,\tau}}{\partial t} + \frac{\partial P^{\$}_{t,\tau}}{\partial r_t}\kappa_r\left[\bar{r} - r_t\right] + \frac{\partial P^{\$}_{t,\tau}}{\partial \pi_t}\kappa_\pi\left[\bar{\pi} - \pi_t\right] \\ & + \frac{1}{2}\frac{\partial^2 P^{\$}_{t,\tau}}{\partial r_t^2}\sigma_r^2 + \frac{1}{2}\frac{\partial^2 P^{\$}_{t,\tau}}{\partial \pi_t^2}\sigma_\pi^2 + \frac{\partial^2 P^{\$}_{t,\tau}}{\partial r_t \partial \pi_t}\rho_{r,\pi}\sigma_r\sigma_\pi. \end{aligned} \tag{A.43}$$

To create a riskless portfolio with nominal bonds we need at least three bonds with different maturities (τ, τ_1, τ_2) to eliminate the sources of randomness. Let us denote the return on a self-financing portfolio V as

$$\frac{dV}{V} = \frac{dP^{\$}_{t,\tau}}{P^{\$}_{t,\tau}} - \omega_1\frac{dP^{\$}_{t,\tau_1}}{P^{\$}_{t,\tau_1}} - \omega_2\frac{dP^{\$}_{t,\tau_2}}{P^{\$}_{t,\tau_2}}, \tag{A.44}$$

where ω_i represents the i-bond's weight on the portfolio. To eliminate the sources of randomness, ω_i's must be chosen such that

$$\underbrace{\begin{bmatrix} \frac{1}{P^{\$}_{t,\tau_1}}\frac{\partial P^{\$}_{t,\tau_1}}{\partial r_t}\sigma_r & \frac{1}{P^{\$}_{t,\tau_2}}\frac{\partial P^{\$}_{t,\tau_2}}{\partial r_t}\sigma_r \\ \frac{1}{P^{\$}_{t,\tau_1}}\frac{\partial P^{\$}_{t,\tau_1}}{\partial \pi_t}\sigma_\pi & \frac{1}{P^{\$}_{t,\tau_2}}\frac{\partial P^{\$}_{t,\tau_2}}{\partial \pi_t}\sigma_\pi \end{bmatrix}}_{H} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \frac{1}{P^{\$}_{t,\tau}} \begin{bmatrix} \frac{\partial P^{\$}_{t,\tau}}{\partial r_t}\sigma_r \\ \frac{\partial P^{\$}_{t,\tau}}{\partial \pi_t}\sigma_\pi \end{bmatrix}, \tag{A.45}$$

which have a solution $(\vec{\omega}^*)$ if the H matrix is non-singular. Then, the instantaneous nominal rate of return of the portfolio, V, is

$$\frac{dV}{V} = \Big[\mu(\tau) - \omega_1^*\mu(\tau_1) - \omega_2^*\mu(\tau_2)\Big]dt. \tag{A.46}$$

To avoid arbitrage opportunities, the riskless portfolio must earn the real risk-free rate of return

$$\begin{aligned}\Big[\mu(\tau) - \omega_1^*\mu(\tau_1) - \omega_2^*\mu(\tau_2) - \pi_t\Big]dt &= r_t dt \\ \mu(\tau) - \pi_t - r_t &= \begin{bmatrix}\mu(\tau_1) & \mu(\tau_2)\end{bmatrix}\begin{bmatrix}\omega_1^* \\ \omega_2^*\end{bmatrix}.\end{aligned} \tag{A.47}$$

Replacing (A.45) in (A.47), we define the market price of risk vector as

$$\begin{bmatrix}\lambda_r & \lambda_\pi\end{bmatrix} = \begin{bmatrix}\mu(\tau_1) & \mu(\tau_2)\end{bmatrix} H^{-1} \tag{A.48}$$

then

$$\mu(\tau) - \pi_t - r_t = \begin{bmatrix}\lambda_r & \lambda_\pi\end{bmatrix}\frac{1}{P^\$_{t,\tau}}\begin{bmatrix}\frac{\partial P^\$_{t,\tau}}{\partial r_t}\sigma_r \\ \frac{\partial P^\$_{t,\tau}}{\partial \pi_t}\sigma_\pi\end{bmatrix}. \tag{A.49}$$

Note that λ_i's are independent of the selection of each maturity so λ_i's are constant across arbitrary maturities. Then, the real excess return of nominal bonds over the real risk-free rate of return is given by the sum of the product of the bond's sensitivity to each source of risk times the market price of risk.

Re-writing (A.49) with (A.43) we get

$$\begin{aligned}&\frac{\partial P^\$_{t,\tau}}{\partial t} + \frac{\partial P^\$_{t,\tau}}{\partial r_t}\kappa_r[\bar{r} - r_t] + \frac{\partial P^\$_{t,\tau}}{\partial \pi_t}\kappa_\pi[\bar{\pi} - \pi_t] - P^\$_{t,\tau}\pi_t - P^\$_{t,\tau}r_t + \frac{1}{2}\frac{\partial^2 P^\$_{t,\tau}}{\partial r_t^2}\sigma_r^2 \\ &\quad + \frac{1}{2}\frac{\partial^2 P^\$_{t,\tau}}{\partial \pi_t^2}\sigma_\pi^2 + \frac{\partial^2 P^\$_{t,\tau}}{\partial r_t \partial \pi_t}\rho_{r,\pi}\sigma_r\sigma_\pi = \lambda_r\frac{\partial P^\$_{t,\tau}}{\partial r_t}\sigma_r + \lambda_\pi\frac{\partial P^\$_{t,\tau}}{\partial \pi_t}\sigma_\pi.\end{aligned} \tag{A.50}$$

Re-ordering terms in (A.50) we find a partial differential equation for arbitrary maturity τ

$$\begin{aligned}0 = -[\pi_t + r_t]P^\$_{t,\tau} &+ \frac{\partial P^\$_{t,\tau}}{\partial t} + \kappa_r[\tilde{r} - r_t]\frac{\partial P^\$_{t,\tau}}{\partial r_t} + \kappa_\pi[\tilde{\pi} - \pi_t]\frac{\partial P^\$_{t,\tau}}{\partial \pi_t} + \frac{\sigma_r^2}{2}\frac{\partial^2 P^\$_{t,\tau}}{\partial r_t^2} \\ &+ \frac{\sigma_\pi^2}{2}\frac{\partial^2 P^\$_{t,\tau}}{\partial \pi_t^2} + \rho_{r,\pi}\sigma_r\sigma_\pi\frac{\partial^2 P^\$_{t,\tau}}{\partial r_t \partial \pi_t},\end{aligned} \tag{A.51}$$

where $\tilde{r} = \bar{r} - \frac{\sigma_r\lambda_r}{\kappa_r}$ and $\tilde{\pi} = \bar{\pi} - \frac{\sigma_\pi\lambda_\pi}{\kappa_\pi}$ are the long-run mean of the real interest rate and inflation under the risk-neutral measure. Considering the boundary condition $P^\$(t, 0, r_t, \pi_t) = 1$ for the bond price function we conjecture a solution to (A.51) with the following form

$$P^{\$}(t, \tau, r_t, \pi_t) = e^{-\left[\bar{A}_{\$}(\tau) + \bar{B}_{\$}(\tau) r_t + \bar{C}_{\$}(\tau)\pi_t\right]}. \tag{A.52}$$

Then,

$$\frac{\partial P^{\$}_{t,\tau}}{\partial t} = \left[\bar{A}'_{\$}(\tau) + \bar{B}'_{\$}(\tau) r_t + \bar{C}'_{\$}(\tau)\pi_t\right] P^{\$}_{t,\tau}, \tag{A.53}$$

$$\frac{\partial P^{\$}_{t,\tau}}{\partial r_t} = -\bar{B}_{\$}(\tau) P^{\$}_{t,\tau}, \tag{A.54}$$

$$\frac{\partial P^{\$}_{t,\tau}}{\partial \pi_t} = -\bar{C}_{\$}(\tau) P^{\$}_{t,\tau}, \tag{A.55}$$

$$\frac{\partial^2 P^{\$}_{t,\tau}}{\partial r_t^2} = \bar{B}_{\$}(\tau)^2 P^{\$}_{t,\tau}, \tag{A.56}$$

$$\frac{\partial^2 P^{\$}_{t,\tau}}{\partial \pi_t^2} = \bar{C}_{\$}(\tau)^2 P^{\$}_{t,\tau}, \tag{A.57}$$

$$\frac{\partial^2 P^{\$}_{t,\tau}}{\partial r_t \partial \pi_t} = \bar{B}_{\$}(\tau)\bar{C}_{\$}(\tau) P^{\$}_{t,\tau}. \tag{A.58}$$

Replacing these derivatives in (A.51) we get

$$\begin{aligned} 0 = \bar{A}'_{\$}(\tau) - \bar{B}_{\$}(\tau)\kappa_r \tilde{r} - \bar{C}_{\$}(\tau)\kappa_\pi \tilde{\pi} + \bar{B}_{\$}(\tau)^2 \frac{\sigma_r^2}{2} + \bar{C}_{\$}(\tau)^2 \frac{\sigma_\pi^2}{2} + \rho_{r,\pi}\sigma_r\sigma_\pi \bar{B}_{\$}(\tau)\bar{C}_{\$}(\tau) \\ + \Big[\bar{B}'_{\$}(\tau) + \kappa_r \bar{B}_{\$}(\tau) - 1\Big] r_t + \Big[\bar{C}'_{\$}(\tau) + \kappa_\pi \bar{C}_{\$}(\tau) - 1\Big]\pi_t. \end{aligned} \tag{A.59}$$

We can reduce (A.59) into a series of ordinary differential equations which can be solved analytically with the boundary conditions $\bar{A}_{\$}(0) = \bar{B}_{\$}(0) = \bar{C}_{\$}(0) = 0$. The solution for $\bar{A}_{\$}(\tau)$, $\bar{B}_{\$}(\tau)$, and $\bar{C}_{\$}(\tau)$ are:

$$\begin{aligned} \bar{A}_{\$}(\tau) = \tilde{r}\Big[\tau - \bar{B}_{\$}(\tau)\Big] + \tilde{\pi}\Big[\tau - \bar{C}_{\$}(\tau)\Big] - \frac{\sigma_r^2}{2\kappa_r^2}\Big[\tau - 2\bar{B}_{\$}(\tau) + \frac{1 - e^{-2\kappa_r\tau}}{2\kappa_r}\Big], \qquad \text{(A.60)} \\ - \frac{\sigma_\pi^2}{2\kappa_\pi^2}\Big[\tau - 2\bar{C}_{\$}(\tau) + \frac{1 - e^{-2\kappa_\pi\tau}}{2\kappa_\pi}\Big] - \frac{\rho_{r,\pi}\sigma_r\sigma_\pi}{\kappa_r\kappa_\pi}\Big[\tau - \bar{B}_{\$}(\tau) - \bar{C}_{\$}(\tau) + \frac{1 - e^{-(\kappa_r+\kappa_\pi)\tau}}{\kappa_r + \kappa_\pi}\Big], \end{aligned}$$

$$\bar{B}_{\$}(\tau) = \frac{1 - e^{-\kappa_r\tau}}{\kappa_r} \tag{A.61}$$

$$\bar{C}_{\$}(\tau) = \frac{1 - e^{-\kappa_\pi\tau}}{\kappa_\pi}. \tag{A.62}$$

A.3.1 A Model with Arbitrageurs

If $\alpha(\tau) = 0$ then there no PHI and mark-to-market investors must clear the market. We can re-write (A.11) as

$$
\begin{aligned}
0 = A'_{\$}(\tau) &- B_{\$}(\tau)\Big[\kappa_r \bar{r} + \gamma\sigma_r\Big[\sigma_r N_r + \rho_{r,\pi}\sigma_\pi N_\pi\Big]\Big] - C_{\$}(\tau)\Big[\kappa_\pi \bar{\pi} + \gamma\sigma_\pi\Big[\sigma_\pi N_\pi + \rho_{r,\pi}\sigma_r N_r\Big]\Big] \\
&+ B_{\$}(\tau)^2\frac{\sigma_r^2}{2} + C_{\$}(\tau)^2\frac{\sigma_\pi^2}{2} + \rho_{r,\pi}\sigma_r\sigma_\pi B_{\$}(\tau)C_{\$}(\tau) \\
&+ \Big[B'_{\$}(\tau) + \kappa_r B_{\$}(\tau) - 1\Big]r_t + \Big[C'_{\$}(\tau) + \kappa_\pi C_{\$}(\tau) - 1\Big]\pi_t, \qquad \text{(A.63)}
\end{aligned}
$$

where

$$
N_r = \int_0^T \Big[s_\tau B(\tau) + s_\tau^{\$} B_{\$}(\tau)\Big]d\tau, \qquad \text{(A.64)}
$$

$$
N_\pi = \int_0^T s_\tau^{\$} C_{\$}(\tau)d\tau. \qquad \text{(A.65)}
$$

Interestingly, (A.63) is equal to (A.59) when

$$
\tilde{r} = \bar{r} + \frac{\sigma_r\gamma\Big[\sigma_r N_r + \rho_{r,\pi}\sigma_\pi N_\pi\Big]}{\kappa_r}, \qquad \text{(A.66)}
$$

$$
\tilde{\pi} = \bar{\pi} + \frac{\sigma_\pi\gamma\Big[\sigma_\pi N_\pi + \rho_{r,\pi}\sigma_r N_r\Big]}{\kappa_\pi}, \qquad \text{(A.67)}
$$

which results in

$$
\lambda_r = -\gamma\Big[\sigma_r N_r + \rho_{r,\pi}\sigma_\pi N_\pi\Big], \qquad \text{(A.68)}
$$

$$
\lambda_\pi = -\gamma\Big[\sigma_\pi N_\pi + \rho_{r,\pi}\sigma_r N_r\Big]. \qquad \text{(A.69)}
$$

Appendix B

Appendix: Market Inflation and Inflation-Linked Bonds

B.1 The Model

B.1.1 The nominal SDF

The nominal SDF, $M_t^{\$}$, is related to the real SDF and the price level index by $M_t^{\$} = M_t Q_t^{-1}$. Then, the dynamic of the nominal SDF is given by

$$\begin{aligned} dM_t^{\$} &= d(M_t Q_t^{-1}) \\ &= Q_t^{-1} dM_t + M_t d(Q_t^{-1}) + dM_t d(Q_t^{-1}). \end{aligned} \tag{B.1}$$

To get $d(Q_t^{-1})$ we apply Itô's lemma and use equation (4.6)

$$d(Q_t^{-1}) = -\frac{1}{Q_t^2} dQ_t + \frac{1}{2}\frac{2}{Q_t^3}(dQ_t)^2 \tag{B.2}$$

$$= -\frac{1}{Q_t}\frac{dQ_t}{Q_t} + \frac{1}{Q_t}\Big[\frac{dQ_t}{Q_t}\Big]^2 \tag{B.3}$$

that results in

$$d(Q_t^{-1})Q_t = -\frac{dQ_t}{Q_t} + \Big[\frac{dQ_t}{Q_t}\Big]^2 \tag{B.4}$$

$$= -\pi_t dt - \sigma_q dZ_{q,t} + \sigma_q^2 dt. \tag{B.5}$$

Dividing equation (B.1) by $M_t^{\$}$ we get

$$\begin{aligned} \frac{dM_t^{\$}}{M_t^{\$}} &= \frac{dM_t}{M_t} + d(Q_t^{-1})Q_t + \frac{dM_t}{M_t} d(Q_t^{-1})Q_t \\ &= -\big[r_t + \pi_t - \sigma_q^2 - \eta_t \sigma_q \rho_{mq}\big] dt - \eta_t dZ_{m,t} - \sigma_q dZ_{q,t}. \end{aligned} \tag{B.6}$$

B.1.2 Real Bonds

Let the price of a zero-coupon real bond with a defined term $\tau = T - t$ be a function of the underlying risk factors

$$P_{t,\tau} = P(t, \tau, \eta_t, r_t). \tag{B.7}$$

We can determine the differential dynamics of the zero-coupon bond price by Itô's lemma

$$dP_{t,\tau} = \frac{\partial P_{t,\tau}}{\partial t} dt + \frac{\partial P_{t,\tau}}{\partial \eta_t} d\eta_t + \frac{\partial P_{t,\tau}}{\partial r_t} dr_t + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2 P_{t,\tau}}{\partial x_{i,t} \partial x_{j,t}} d\langle x_i, x_j \rangle (t) \tag{B.8}$$

where $x = (\eta, r)$ is the set of state variables and $d\langle x_i, x_j\rangle(t)$ represent the quadratic covariation between $x_{i,t}$ and $x_{j,t}$. Using (4.3)-(4.2) and (B.8) we get

$$dP_{t,\tau} = \mu(\tau)\, dt + \frac{\partial P_{t,\tau}}{\partial \eta_t} \sigma_\eta dZ_{\eta,t} + \frac{\partial P_{t,\tau}}{\partial r_t} \sigma_r dZ_{r,t} \tag{B.9}$$

where

$$\mu(\tau) = \frac{\partial P_{t,\tau}}{\partial t} + \frac{\partial P_{t,\tau}}{\partial \eta_t} \kappa_\eta (\bar{\eta} - \eta_t) + \frac{\partial P_{t,\tau}}{\partial r_t} \kappa_r (\bar{r} - r_t) + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2 P_{t,\tau}}{\partial x_{i,t} \partial x_{j,t}} \rho_{x_i x_j} \sigma_{x_i} \sigma_{x_j}. \tag{B.10}$$

Considering the boundary condition $P(t, 0, \eta_t, r_t) = 1$ for the bond price function, we conjecture equation (4.12) is a solution to (4.11) which results that the non-arbitrage condition can be re-express as

$$\begin{aligned} & A'(\tau) - B(\tau)\kappa_\eta \bar{\eta} - C(\tau)\kappa_r \bar{r} + \frac{\sigma_\eta^2}{2} B(\tau)^2 + \frac{\sigma_r^2}{2} C(\tau)^2 + B(\tau)C(\tau)\sigma_\eta \sigma_r \rho_{\eta r} \\ & \quad + \Big[B'(\tau) + B(\tau)\kappa_\eta + B(\tau)\sigma_\eta \rho_{m\eta} + C(\tau)\sigma_r \rho_{mr}\Big]\eta_t + \Big[C'(\tau) + C(\tau)\kappa_r - 1\Big] r_t = 0 \end{aligned} \tag{B.11}$$

which is a PDE that should be satisfied for all values of η_t, and r_t which holds if every expression is zero. Equating each term to zero, results in four ordinary differential equations (ODEs) with a boundary condition $P_{t,0} = 1 \Rightarrow A(0) = B(0) = C(0) = 0$. Then, the solution is given by:

$$A(\tau) = \sum_i a_i \tag{B.12}$$

$$B(\tau) = -\frac{\sigma_r \rho_{mr}}{\kappa_r \kappa_\eta^*} + e^{-\kappa_\eta^* \tau} + \frac{\sigma_r \rho_{mr}}{(\kappa_\eta^* - \kappa_r)\kappa_r} e^{-\kappa_r \tau} \tag{B.13}$$

$$C(\tau) = \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} \tag{B.14}$$

where

$$\kappa_\eta^* = \kappa_\eta + \rho_{m\eta}\sigma_\eta \tag{B.15}$$

and $a_i = a_{n,i}$ (Appendix B.1.3) in which σ_q and all the parameters of the expected inflation processes are zero.

B.1.3 Nominal Bonds

Let the price of a zero-coupon nominal bond with a defined term $\tau = T - t$ be a function of the underlying risk factors

$$P_{t,\tau}^{\$} = P^{\$}(t, \tau, \eta_t, r_t, \pi_t). \tag{B.16}$$

We can determine the differential dynamics of the zero-coupon nominal bond price by Itô's lemma

$$dP_{t,\tau}^{\$} = \frac{\partial P_{t,\tau}^{\$}}{\partial t}dt + \frac{\partial P_{t,\tau}^{\$}}{\partial \eta_t}d\eta_t + \frac{\partial P_{t,\tau}^{\$}}{\partial r_t}dr_t + \frac{\partial P_{t,\tau}^{\$}}{\partial \pi_t}d\pi_t + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 P_{t,\tau}}{\partial x_{i,t}\partial x_{j,t}}d\langle x_i, x_j\rangle(t) \tag{B.17}$$

where $x = (\eta, r, \pi)$ is the set of state variables and $d\langle x_i, x_j\rangle(t)$ represent the quadratic covariation between $x_{i,t}$ and $x_{j,t}$. Given (4.7) and (4.8) the dynamics of the expected rate of inflation is given by

$$d\pi_t = \sum_{j=c,e,f} \omega_j \Big(\kappa_{\pi_j}[\bar{\pi}_j - \pi_{j,t}]dt + \sigma_{\pi_j} dZ_{\pi_j,t}\Big). \tag{B.18}$$

Then, using (4.2), (4.3), (B.17) and (B.18) we get

$$dP_{t,\tau}^{\$} = \mu_{\$}(\tau)\, dt + \frac{\partial P_{t,\tau}^{\$}}{\partial \eta_t}\sigma_\eta dZ_{\eta,t} + \frac{\partial P_{t,\tau}^{\$}}{\partial r_t}\sigma_r dZ_{r,t} + \frac{\partial P_{t,\tau}^{\$}}{\partial \pi_t}\boldsymbol{\sigma}_\pi' d\mathbf{Z}_{\pi,t} \tag{B.19}$$

where

$$\begin{aligned}\mu_{\$}(\tau) =& \frac{\partial P_{t,\tau}^{\$}}{\partial t} + \frac{\partial P_{t,\tau}^{\$}}{\partial \eta_t}\kappa_\eta[\bar{\eta} - \eta_t] + \frac{\partial P_{t,\tau}^{\$}}{\partial r_t}\kappa_r[\bar{r} - r_t] + \frac{\partial P_{t,\tau}^{\$}}{\partial \pi_t}\sum_{j=c,e,f}\omega_j\kappa_{\pi_j}[\bar{\pi}_j - \pi_{j,t}] \\ &+ \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{\partial^2 P_{t,\tau}^{\$}}{\partial x_{i,t}\partial x_{j,t}}\rho_{x_i x_j}\sigma_{x_i}\sigma_{x_j},\end{aligned} \tag{B.20}$$

$\boldsymbol{\sigma}_\pi' = [\omega_c\sigma_c, \omega_e\sigma_e, \omega_f\sigma_f]$, and $d\mathbf{Z}_{\pi,t}' = [dZ_{\pi_c,t}, dZ_{\pi_e,t}, dZ_{\pi_f,t}]$. Considering the boundary condition $P^{\$}(t, 0, \eta_t, r_t, \pi_t) = 1$ for the nominal bond price function, we conjecture equation (4.18) is a solution to (4.17) which results that the non-arbitrage condition can be re-express as

$$\begin{aligned}
&A'_n(\tau) - B_n(\tau)\kappa_\eta\bar{\eta} - C_n(\tau)\kappa_r\bar{r} - \sum_{j=c,e,f} D_{n,j}(\tau)\omega_j\kappa_{\pi_j}\bar{\pi}_j \\
&\quad + \frac{\sigma_\eta^2}{2}B_n(\tau)^2 + \frac{\sigma_r^2}{2}C_n(\tau)^2 + \sum_{j=c,e,f}\frac{\omega_j^2\sigma_{\pi_j}^2}{2}D_{n,j}(\tau)^2 \\
&\quad + B_n(\tau)C_n(\tau)\sigma_\eta\sigma_r\rho_{\eta r} + \sum_{j=c,e,f} B_n(\tau)D_{n,j}(\tau)\sigma_\eta\omega_j\sigma_{\pi_j}\rho_{\eta\pi_j} \\
&\quad + \sum_{j=c,e,f} C_n(\tau)D_{n,j}(\tau)\sigma_r\omega_j\sigma_{\pi_j}\rho_{r\pi_j} + \sum_{\substack{j=c,e,f\\ k\neq j}} D_{n,k}(\tau)D_{n,j}(\tau)\omega_k\omega_j\sigma_{\pi_k}\sigma_{\pi_j}\rho_{\pi_k\pi_j} \\
&\quad + B_n(\tau)\sigma_\eta\sigma_q\rho_{q\eta} + C_n(\tau)\sigma_r\sigma_q\rho_{qr} + \sum_{j=c,e,f} D_{n,j}(\tau)\omega_j\sigma_{\pi_j}\sigma_q\rho_{q\pi_j} + \sigma_q^2 \\
&\quad + \Big[B'_n(\tau) + B_n(\tau)[\kappa_\eta + \sigma_\eta\rho_{m\eta}] + C_n(\tau)\sigma_r\rho_{mr} + \sum_{j=c,e,f} D_{n,j}(\tau)\omega_j\sigma_{\pi_j}\rho_{m\pi_j} + \sigma_q\rho_{mq}\Big]\eta_t \\
&\quad + \Big[C'_n(\tau) + C_n(\tau)\kappa_r - 1\Big]r_t + \sum_{j=c,e,f}\Big[D'_{n,j}(\tau) + D_{n,j}(\tau)\kappa_{\pi_j} - 1\Big]\omega_j\pi_{j,t} = 0
\end{aligned} \tag{B.21}$$

which is a PDE that should be satisfied for all values of η_t, r_t, $\pi_{j,t}$ which holds if every expression is zero. Equating each term to zero, results in four ordinary differential equations (ODEs) with a boundary condition $P^{\$}_{t,0} = 1 \Rightarrow A_n(0) = B_n(0) = C_n(0) = D_{n,j}(0) = 0$. Then, the solution is given by:

$$A_n(\tau) = \sum_i a_{n,i} \tag{B.22}$$

$$B_n(\tau) = b_1 + b_2 e^{-\kappa_\eta^*\tau} + b_3 e^{-\kappa_r\tau} + \sum_{j=c,e,f} b_{4,j}e^{-\kappa_{\pi_j}\tau} \tag{B.23}$$

$$C_n(\tau) = \frac{1 - e^{-\kappa_r\tau}}{\kappa_r} \tag{B.24}$$

$$D_{n,j}(\tau) = \frac{1 - e^{-\kappa_{\pi_j}\tau}}{\kappa_{\pi_j}} \qquad \text{for } j = c, e, f \tag{B.25}$$

where

$$\kappa_\eta^* = \kappa_\eta + \rho_{m\eta}\sigma_\eta \tag{B.26}$$

$$b_1 = -\frac{\sigma_q\rho_{mq}}{\kappa_\eta^*} - \frac{\sigma_r\rho_{mr}}{\kappa_\eta^*\kappa_r} - \sum_{j=c,e,f}\frac{\omega_j\sigma_{\pi_j}\rho_{m\pi_j}}{\kappa_\eta^*\kappa_{\pi_j}} \tag{B.27}$$

$$b_2 = -b_1 - b_3 - \sum_{j=c,e,f} b_{4,j} \tag{B.28}$$

$$b_3 = \frac{\sigma_r\rho_{mr}}{(\kappa_{\eta^*} - \kappa_r)\kappa_r} \tag{B.29}$$

$$b_{4,j} = \frac{\omega_j\sigma_{\pi_j}\rho_{m\pi_j}}{(\kappa_{\eta^*} - \kappa_{\pi_j})\kappa_{\pi_j}} \tag{B.30}$$

$$\int_t^T A_n'(T-s)ds = A_n(\tau) = \sum_i a_{n,i} \tag{B.31}$$

$$\begin{aligned} a_{n,1} &= \kappa_\eta \bar{\eta} \int_t^T B_n(T-s)\, ds \\ &= \kappa_\eta \bar{\eta} \Big[b_1 \tau + b_2 \frac{1-e^{-\kappa_\eta^* \tau}}{\kappa_\eta^*} + b_3 \frac{1-e^{-\kappa_r \tau}}{\kappa_r} + \sum_{j=c,e,f} b_{4,j} \frac{1-e^{-\kappa_{\pi_j} \tau}}{\kappa_{\pi_j}} \Big]. \end{aligned} \tag{B.32}$$

$$\begin{aligned} a_{n,2} &= \kappa_r \bar{r} \int_t^T C_n(T-s) ds \\ &= \bar{r} \int_t^T \Big[1 - e^{-\kappa_r (T-s)} \Big] ds \\ &= \bar{r} \Big[\tau - C_n(\tau) \Big]. \end{aligned} \tag{B.33}$$

$$\begin{aligned} a_{n,3} &= \sum_{j=c,e,f} \Big[\omega_j \kappa_{\pi_j} \bar{\pi}_j \int_t^T D_{n,j}(T-s) ds \Big] \\ &= \sum_{j=c,e,f} \Big[\omega_j \bar{\pi}_j \int_t^T \Big[1 - e^{-\kappa_{\pi_j}(T-s)} \Big] ds \Big] \\ &= \sum_{j=c,e,f} \omega_j \bar{\pi}_j \Big[\tau - D_{n,j}(\tau) \Big]. \end{aligned} \tag{B.34}$$

$$\begin{aligned} -a_{n,4} &= \frac{\sigma_\eta^2}{2} \int_t^T B_n(T-s)^2 ds \\ &= \frac{\sigma_\eta^2}{2} \Big[b_1^2 \tau + 2b_1 b_2 \frac{1-e^{-\kappa_\eta^* \tau}}{\kappa_\eta^*} + 2b_1 b_3 \frac{1-e^{-\kappa_r \tau}}{\kappa_r} + \sum_{j=c,e,f} 2b_1 b_{4,j} \frac{1-e^{-\kappa_{\pi_j} \tau}}{\kappa_{\pi_j}} \\ &\qquad + b_2^2 \frac{1-e^{-2\kappa_\eta^* \tau}}{2\kappa_\eta^*} + 2b_2 b_3 \frac{1-e^{-(\kappa_\eta^* + \kappa_r)\tau}}{\kappa_\eta^* + \kappa_r} + \sum_{j=c,e,f} 2b_2 b_{4,j} \frac{1-e^{-(\kappa_\eta^* + \kappa_{\pi_j})\tau}}{\kappa_\eta^* + \kappa_{\pi_j}} \\ &\qquad + b_3^2 \frac{1-e^{-2\kappa_r \tau}}{2\kappa_r} + \sum_{j=c,e,f} 2b_3 b_{4,j} \frac{1-e^{-(\kappa_r + \kappa_{\pi_j})\tau}}{\kappa_r + \kappa_{\pi_j}} + \sum_{j=c,e,f} b_{4,j}^2 \frac{1-e^{-2\kappa_{\pi_j} \tau}}{2\kappa_{\pi_j}} \\ &\qquad + \sum_{\substack{j=c,e,f \\ k \neq j}} 2b_{4,j} b_{4,k} \frac{1-e^{-(\kappa_{\pi_j} + \kappa_{\pi_k})\tau}}{\kappa_{\pi_j} + \kappa_{\pi_k}} \Big]. \end{aligned} \tag{B.35}$$

$$\begin{aligned} -a_{n,5} &= \frac{\sigma_r^2}{2} \int_t^T C_n(T-s)^2 ds \\ &= \frac{\sigma_r^2}{2\kappa_r^2} \int_t^T \Big[1 - e^{-\kappa_r (T-s)} \Big]^2 ds \\ &= \frac{\sigma_r^2}{2\kappa_r^2} \int_t^T \Big[1 - 2e^{-\kappa_r (T-s)} + e^{-2\kappa_r (T-s)} \Big] ds \\ &= \frac{\sigma_r^2}{2\kappa_r^2} \Big[\tau - 2C_n(\tau) + \frac{1-e^{-2\kappa_r \tau}}{2\kappa_r} \Big]. \end{aligned} \tag{B.36}$$

$$
\begin{aligned}
-a_{n,6} &= \sum_{j=c,e,f}\left[\frac{\omega_j^2\sigma_{\pi_j}^2}{2}\int_t^T D_{n,j}(T-s)^2 ds\right] \\
&= \sum_{j=c,e,f}\left[\frac{\omega_j^2\sigma_{\pi_j}^2}{2\kappa_{\pi_j}^2}\int_t^T \left[1-e^{-\kappa_{\pi_j}(T-s)}\right]^2 ds\right] \\
&= \sum_{j=c,e,f}\left[\frac{\omega_j^2\sigma_{\pi_j}^2}{2\kappa_{\pi_j}^2}\int_t^T \left[1-2e^{-\kappa_{\pi_j}(T-s)}+e^{-2\kappa_{\pi_j}(T-s)}\right] ds\right] \\
&= \sum_{j=c,e,f}\frac{\omega_j^2\sigma_{\pi_j}^2}{2\kappa_{\pi_j}^2}\left[\tau-2D_{n,j}(\tau)+\frac{1-e^{-2\kappa_{\pi_j}\tau}}{2\kappa_{\pi_j}}\right]. \qquad \text{(B.37)}
\end{aligned}
$$

$$
\begin{aligned}
-a_{n,7} &= \sigma_\eta\sigma_r\rho_{\eta r}\int_t^T B_n(T-s)C_n(T-s)ds \\
&= \frac{\sigma_\eta\sigma_r\rho_{\eta r}}{\kappa_r}\left[b_1\tau+b_2\frac{1-e^{-\kappa_\eta^*\tau}}{\kappa_\eta^*}+b_3\frac{1-e^{-\kappa_r\tau}}{\kappa_r}+\sum_{j=c,e,f}b_{4,j}\frac{1-e^{-\kappa_{\pi_j}\tau}}{\kappa_{\pi_j}}\right] \qquad \text{(B.38)} \\
&\quad -\frac{\sigma_\eta\sigma_r\rho_{\eta r}}{\kappa_r}\left[b_1\frac{1-e^{-\kappa_r\tau}}{\kappa_r}+b_2\frac{1-e^{-(\kappa_\eta^*+\kappa_r)\tau}}{\kappa_\eta^*+\kappa_r}+b_3\frac{1-e^{-2\kappa_r\tau}}{2\kappa_r}+\sum_{j=c,e,f}b_{4,j}\frac{1-e^{-(\kappa_{\pi_j}+\kappa_r)\tau}}{\kappa_{\pi_j}+\kappa_r}\right].
\end{aligned}
$$

$$
\begin{aligned}
-a_{n,8} &= \sum_{k=c,e,f}\left[\sigma_\eta\omega_k\sigma_{\pi_k}\rho_{\eta\pi_k}\int_t^T B_n(T-s)D_{n,k}(T-s)ds\right] \\
&= \sum_{k=c,e,f}\frac{\sigma_\eta\omega_k\sigma_{\pi_k}\rho_{\eta\pi_k}}{\kappa_{\pi_k}}\left[b_1\tau+b_2\frac{1-e^{-\kappa_\eta^*\tau}}{\kappa_\eta^*}+b_3\frac{1-e^{-\kappa_r\tau}}{\kappa_r}+\sum_{j=c,e,f}b_{4,j}\frac{1-e^{-\kappa_{\pi_j}\tau}}{\kappa_{\pi_j}}\right] \qquad \text{(B.39)} \\
&\quad -\sum_{k=c,e,f}\frac{\sigma_\eta\omega_k\sigma_{\pi_k}\rho_{\eta\pi_k}}{\kappa_{\pi_k}}\left[b_1\frac{1-e^{-\kappa_{\pi_k}\tau}}{\kappa_{\pi_k}}+b_2\frac{1-e^{-(\kappa_\eta^*+\kappa_{\pi_k})\tau}}{\kappa_\eta^*+\kappa_{\pi_k}}+b_3\frac{1-e^{-(\kappa_r+\kappa_{\pi_k})\tau}}{\kappa_r+\kappa_{\pi_k}}+\sum_{j=c,e,f}b_{4,j}\frac{1-e^{-(\kappa_{\pi_j}+\kappa_{\pi_k})\tau}}{\kappa_{\pi_j}+\kappa_{\pi_k}}\right].
\end{aligned}
$$

$$
\begin{aligned}
-a_{n,9} &= \sum_{j=c,e,f}\left[\sigma_r\omega_j\sigma_{\pi_j}\rho_{r\pi_j}\int_t^T C_n(T-s)D_{n,j}(T-s)ds\right] \\
&= \sum_{j=c,e,f}\frac{\sigma_r\omega_j\sigma_{\pi_j}\rho_{r\pi_j}}{\kappa_r\kappa_{\pi_j}}\left[\tau-C_n(\tau)-D_{n,j}(\tau)+\frac{1-e^{-(\kappa_r+\kappa_{\pi_j})\tau}}{\kappa_r+\kappa_{\pi_j}}\right]. \qquad \text{(B.40)}
\end{aligned}
$$

$$
\begin{aligned}
-a_{n,10} &= \sum_j\sum_{\substack{k=c,e,f\\k\neq j}}\left[\omega_k\omega_j\sigma_{\pi_k}\sigma_{\pi_j}\rho_{\pi_k\pi_j}\int_t^T D_{n,k}(T-s)D_{n,j}(T-s)ds\right] \\
&= \sum_j\sum_{\substack{k=c,e,f\\k\neq j}}\frac{\omega_k\omega_j\sigma_{\pi_k}\sigma_{\pi_j}\rho_{\pi_k\pi_j}}{\kappa_{\pi_k}\kappa_{\pi_j}}\left[\tau-D_{n,k}(\tau)-D_{n,j}(\tau)+\frac{1-e^{-(\kappa_{\pi_k}+\kappa_{\pi_j})\tau}}{\kappa_{\pi_k}+\kappa_{\pi_j}}\right]. \qquad \text{(B.41)}
\end{aligned}
$$

$$
\begin{aligned}
-a_{n,11} &= \sigma_\eta\sigma_q\rho_{q\eta}\int_t^T B_n(T-s)ds \\
&= \sigma_\eta\sigma_q\rho_{q\eta}\left[b_1\tau+b_2\frac{1-e^{-\kappa_\eta^*\tau}}{\kappa_\eta^*}+b_3\frac{1-e^{-\kappa_r\tau}}{\kappa_r}+\sum_{j=c,e,f}b_{4,j}\frac{1-e^{-\kappa_{\pi_j}\tau}}{\kappa_{\pi_j}}\right]. \qquad \text{(B.42)}
\end{aligned}
$$

$$
\begin{aligned}
-a_{n,12} &= \sigma_r\sigma_q\rho_{qr}\int_t^T C_n(T-s)ds \\
&= \frac{\sigma_r\sigma_q\rho_{qr}}{\kappa_r}\left[\tau-C_n(\tau)\right]. \qquad \text{(B.43)}
\end{aligned}
$$

$$
\begin{aligned}
-a_{n,13} &= \sum_{j=c,e,f} \left[\omega_j \sigma_{\pi_j} \sigma_q \rho_{q\pi_j} \int_t^T D_{n,j}(T-s) ds \right] \\
&= \sum_{j=c,e,f} \frac{\omega_j \sigma_{\pi_j} \sigma_q \rho_{q\pi_j}}{\kappa_{\pi_j}} \left[\tau - D_{n,j}(\tau) \right].
\end{aligned}
\tag{B.44}
$$

$$
a_{n,14} = \int_t^T \sigma_q^2 ds = \sigma_q^2 \tau.
\tag{B.45}
$$

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